

# On the range of exponential functionals of Lévy processes

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## Abstract

We characterize the support of the law of the exponential functional  $\int_0^\infty e^{-\xi s} d\eta_s$  of two independent Lévy processes  $\xi$  and  $\eta$ . Further, we study the range of the mapping  $\Phi_\xi$  for a fixed Lévy process  $\xi$ , which maps the law of  $\eta_1$  to the law of the corresponding exponential functional  $\int_0^\infty e^{-\xi s} d\eta_s$ , where  $(\eta_t)_{t \geq 0}$  is a one-dimensional Lévy process, independent of  $\xi$ . It is shown that the range of this mapping is closed under weak convergence and in the special case of positive distributions several characterizations of laws in the range are given.

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## 1 Introduction

Given a bivariate Lévy process  $(\xi, \eta)^T = ((\xi_t, \eta_t)^T)_{t \geq 0}$ , its *exponential functional* is defined as

$$V := \int_0^\infty e^{-\xi s} d\eta_s, \quad (1.1)$$

provided that the integral converges almost surely. Exponential functionals of Lévy processes appear as stationary distributions of generalized Ornstein-Uhlenbeck (GOU) processes. In particular, if  $\xi$  and  $\eta$  are independent and  $\xi_t$  tends to  $+\infty$  as  $t \rightarrow \infty$  almost

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surely, then the law of  $V$  defined in (1.1) is the stationary distribution of the GOU process

$$V_t = e^{-\xi_t} \left( \int_0^t e^{\xi_{s-}} d\eta_s + V_0 \right), \quad t \geq 0, \quad (1.2)$$

where  $V_0$  is a starting random variable, independent of  $(\xi, \eta)^T$ , on the same probability space (cf. [22, Theorem 2.1]). Hence, when  $V_0$  is chosen to have the same distribution as  $V$ , then the process  $(V_t)_{t \geq 0}$  is strictly stationary.

Unless  $\xi_t = at$  with  $a > 0$ , the distribution of  $V$  is known only in a few special cases. See e.g. Bertoin and Yor [8] for a survey on exponential functionals of the form  $V = \int_0^\infty e^{-\xi_{s-}} ds$ , or Gjessing and Paulsen [15], who determine the distribution of  $\int_0^\infty e^{-\xi_{s-}} d\eta_s$  for some cases. We state the following example due to Dufresne (e.g. [8, Equation (16)]) of an exponential functional whose distribution has been determined and to which we will refer later. Here and in the following we write “ $\stackrel{d}{=}$ ” to denote equality in distribution of random variables.

**Example 1.1.** For  $(\xi_t, \eta_t) = (\sigma B_t + at, t)$  with  $\sigma > 0$ ,  $a > 0$  and a standard Brownian motion  $(B_t)_{t \geq 0}$  it holds

$$V = \int_0^\infty e^{-(\sigma B_t + at)} dt \stackrel{d}{=} \frac{2}{\sigma^2 \Gamma_{\frac{2a}{\sigma^2}}},$$

where  $\Gamma_r$  denotes a standard Gamma random variable with shape parameter  $r$ , i.e. with density

$$P(\Gamma_r \in dx) = \frac{x^{r-1}}{\Gamma(r)} e^{-x} \mathbf{1}_{(0, \infty)}(x) dx.$$

Denote by  $\mathcal{L}(X)$  the law of a random variable  $X$  and let  $\xi = (\xi_t)_{t \geq 0}$  be a one-dimensional Lévy process drifting to  $+\infty$ . In this paper we will consider the mapping

$$\begin{aligned} \Phi_\xi : D_\xi &\rightarrow \mathcal{P}(\mathbb{R}) := \text{the set of probability distributions on } \mathbb{R}, \\ \mathcal{L}(\eta_1) &\mapsto \mathcal{L} \left( \int_0^\infty e^{-\xi_{s-}} d\eta_s \right), \end{aligned}$$

defined on

$$\begin{aligned} D_\xi &:= \{ \mathcal{L}(\eta_1) : \eta = (\eta_t)_{t \geq 0} \text{ one-dimensional Lévy process independent of } \xi \\ &\quad \text{such that } \int_0^\infty e^{-\xi_{s-}} d\eta_s \text{ converges a.s.} \}. \end{aligned}$$

An explicit description of  $D_\xi$  in terms of the characteristic triplets (cf. (1.3)) of  $\xi$  and  $\eta$  follows from Theorem 2 in Erickson and Maller [14]. Denote the range of  $\Phi_\xi$  by

$$R_\xi := \Phi_\xi(D_\xi).$$

Although the domain  $\Phi_\xi$  can be completely characterized by [14], much less is known about the range  $R_\xi$  and properties of the mapping  $\Phi_\xi$ . In the case that  $\xi_t = at$ ,  $a > 0$  is deterministic, it is well known that  $D_\xi = \text{ID}_{\log}(\mathbb{R})$ , the set of real-valued infinitely

divisible distributions with finite  $\log^+$ -moment, and that  $\Phi_\xi$  is an algebraic isomorphism between  $\text{ID}_{\log}(\mathbb{R})$  and  $R_\xi = L(\mathbb{R})$ , the set of real-valued selfdecomposable distributions [17, Proposition 3.6.10].

For general  $\xi$ , the mapping  $\Phi_\xi$  has already been studied in [5], where it has been shown that  $\Phi_\xi$  is injective in many cases, while injectivity cannot be obtained if  $\xi$  and  $\eta$  are allowed to exhibit a dependence structure. Further in [5] conditions for continuity (in a weak sense) of  $\Phi_\xi$  are given. These results were then used to obtain some information on the range  $R_\xi$ . In particular it has been shown that centered Gaussian distributions can only be obtained in the setting of (classical) OU processes, namely, for  $\xi$  being deterministic and  $\eta$  being a Brownian motion.

In this paper we take up the subject of studying properties of the mapping  $\Phi_\xi$  and of distributions in  $R_\xi$ , and start in Section 2 with a classification of possible supports of the laws in  $R_\xi$ . Section 3 is devoted to show closedness of the range  $R_\xi$  under weak convergence. It also follows that the inverse mapping  $\Phi_\xi^{-1}$  is continuous if it is well-defined, i.e. if  $\Phi_\xi$  is injective. In Sections 4 and 5 we specialize on positive distributions in  $R_\xi$ . Section 4 gives a general criterion for positive distributions to belong to  $R_\xi$ . In Section 5 we use this criterion to obtain further results in the case that  $\xi$  is a Brownian motion with drift. We derive a differential equation for the Laplace exponent of a positive distribution in  $R_\xi$  and from this we gain concrete conditions in terms of Lévy measure and drift for some distributions to be in  $R_\xi$ . We end up studying the special case of positive stable distributions in  $R_\xi$ .

For an  $\mathbb{R}^d$ -valued Lévy process  $X = (X_t)_{t \geq 0}$ , the *characteristic exponent* is given by its Lévy-Khintchine formula (e.g. [26, Theorem 8.1])

$$\begin{aligned} \log \phi_X(u) &:= \log E \left[ e^{i\langle u, X_1 \rangle} \right] \\ &= i\langle \gamma_X, u \rangle - \frac{1}{2}\langle u, A_X u \rangle + \int_{\mathbb{R}^d} (e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle \mathbb{1}_{|x| \leq 1}) \nu_X(dx), \quad u \in \mathbb{R}, \end{aligned} \tag{1.3}$$

where  $(\gamma_X, A_X, \nu_X)$  is the *characteristic triplet* of the Lévy process  $X$ . In case that  $X$  is real-valued we will usually replace  $A_X$  by  $\sigma_X^2$ . We set  $\nu_X(\{0\}) = 0$  for any Lévy measure  $\nu_X$ . In the special case of subordinators in  $\mathbb{R}$ , i.e. nondecreasing Lévy processes, we will also use the Laplace transform

$$\mathbb{L}_X(u) := E[e^{-uX_1}] = e^{\psi_X(u)}, \quad u \geq 0,$$

of  $X$  and call  $\psi_X(u)$  the Laplace exponent of the Lévy process  $X$ . We refer to [26] for further information on Lévy processes. In the following, when the symbol  $X$  is regarded as a real-valued random variable, we also use the notation  $\phi_X(u)$  and  $\mathbb{L}_X(u)$  for its characteristic function and Laplace transform, respectively. The Fourier transform of a finite measure  $\mu$  on  $\mathbb{R}$  is written as  $\hat{\mu}(u) = \int_{\mathbb{R}} e^{iux} \mu(dx)$ . We write “ $\xrightarrow{d}$ ” to denote convergence in distribution of random variables, and “ $\xrightarrow{w}$ ” to denote weak convergence of probability measures. We use the abbreviation “i.i.d.” for “independent and identically distributed”. The set of all twice continuously differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  which are bounded will be denoted by  $C_b^2(\mathbb{R})$ , and the subset of all  $f : \mathbb{R} \rightarrow \mathbb{R}$  which have additionally compact support by  $C_c^2(\mathbb{R})$ .

## 2 On the support of the exponential functional

In this section we shall give the support of the distribution of the exponential functional  $V = \int_0^\infty e^{-\xi_{s-}} d\eta_s$  when  $\xi$  and  $\eta$  are independent Lévy processes. In particular it turns out that the support will always be a closed interval. A similar result does not hold for solutions of arbitrary random recurrence equations, or for exponential functionals of Lévy processes with dependent  $\xi$  and  $\eta$ , as we shall show in Remark 2.4.

For  $\xi$  being spectrally negative, it is well known (e.g. [7]) that  $V$  has a selfdecomposable and hence infinitely divisible distribution. In [26, Theorem 24.10] a characterization of the support of infinitely divisible distributions is given in terms of the Lévy triplet. In particular the support of a selfdecomposable distribution on  $\mathbb{R}$  is either a single point,  $\mathbb{R}$  itself or a one-sided unbounded interval. Unfortunately the characteristic triplet of  $V$  is not known in general and also, for not spectrally negative  $\xi$  this result can not be applied.

Before we characterize the support of the law of  $V = \int_0^\infty e^{-\xi_{s-}} d\eta_s$  when  $\xi$  and  $\eta$  are general independent Lévy processes, we treat the special case when  $\eta_t = t$  in the following lemma. Much attention has been paid to this case, and in particular, it has been shown that the stationary solution has a density under various conditions, see e.g. Pardo et al. [24] or Carmona et al. [12]. Haas and Rivero [16, Theorem 1.4, Lemma 2.1] gave a characterization when this law is bounded and obtained that this is the case if and only if  $\xi$  is a subordinator with strictly positive drift, and derived the support then. So parts of the following lemma follow already from results in [16], nevertheless we have decided to give a detailed proof.

**Lemma 2.1.** *Let  $\xi$  be a Lévy process drifting to  $+\infty$  and set  $V = \int_0^\infty e^{-\xi_s} ds$ . Then*

$$\text{supp } \mathcal{L}(V) = \begin{cases} \left\{ \frac{1}{b} \right\}, & \text{if } \xi_t = bt \text{ with } b > 0, \\ [0, \frac{1}{b}], & \text{if } \xi \text{ is a non-deterministic subordinator with drift } b > 0, \\ [\frac{1}{b}, \infty), & \text{if } \xi \text{ is non-deterministic and of finite variation,} \\ & \text{with drift } b > 0 \text{ and } \nu_\xi((0, \infty)) = 0, \\ [0, \infty), & \text{otherwise.} \end{cases}$$

*Proof.* The claim is clear if  $\xi$  is deterministic, while it follows from Remark 1.1 if  $\xi$  is a Brownian motion with drift, so suppose that  $\nu_\xi \not\equiv 0$ . Suppose first that  $\nu_\xi((0, \infty)) > 0$ , and let  $x_0 \in \text{supp } \mathcal{L}(V) \cap (0, \infty)$ . Let  $c \in \text{supp } \nu_\xi \cap (0, \infty)$  and  $y_0 \in (e^{-c}x_0, x_0)$ . We shall show that also  $y_0 \in \text{supp } \mathcal{L}(V)$ , so that by induction  $\text{supp } \mathcal{L}(V)$  must be an interval with lower endpoint 0 if  $\nu_\xi((0, \infty)) > 0$ . To see this, define  $z_0 \in (0, y_0)$  so that

$$z_0 + e^{-c}(x_0 - z_0) = y_0.$$

Let  $\varepsilon \in (0, \frac{x_0 - z_0}{2})$  and define

$$A = A_\varepsilon := \left\{ \omega \in \Omega : \int_0^\infty e^{-\xi_s(\omega)} ds \in (x_0 - \varepsilon, x_0 + \varepsilon) \right\}.$$

Then  $P(A) > 0$  since  $x_0 \in \text{supp } \mathcal{L}(V)$ . Define the stopping time  $T_1 \in [0, \infty]$  by

$$T_1 := \inf \left\{ t \geq 0 : \int_0^t e^{-\xi_s(\omega)} ds = z_0 \right\}.$$

Since  $t \mapsto \int_0^t e^{-\xi_s(\omega)} ds$  is continuous,  $T_1$  is finite on  $A$ . Let  $\delta_1 \in (0, \frac{x_0 - z_0}{2})$  and  $\delta_2 \in (0, c)$ . Then  $\nu_\eta((c - \delta_2, c + \delta_2)) > 0$ , and since  $P(A) > 0$ , there are a (sufficiently large) constant  $K = K(\varepsilon, \delta_1, \delta_2) > 0$  and a (sufficiently small) constant  $\delta = \delta(\varepsilon, \delta_1, \delta_2) > 0$  such that  $\delta < 1$  and

$$B := B_{\varepsilon, \delta_1, \delta_2, \delta, K} := A \cap \left\{ T_1 \leq K, \int_{T_1}^{T_1 + \delta} e^{-\xi_s} ds \leq \delta_1, \right. \\ \left. \Delta \xi_s \notin (c - \delta_2, c + \delta_2), \forall s \in (T_1, T_1 + \delta] \right\}$$

has a positive probability. Now define the set  $C = C_{\varepsilon, \delta_1, \delta_2, \delta, K}$  to be the set of all  $\omega \in \Omega$ , for which there exists an  $\omega' \in B$ , some time  $t(\omega') \in (T_1 \wedge K, (T_1 \wedge K) + \delta]$  and some  $\alpha(\omega') \in (c - \delta_2, c + \delta_2)$  such that

$$(\xi_t(\omega))_{t \geq 0} = (\xi_t(\omega') + \alpha(\omega') \mathbb{1}_{[t(\omega'), \infty)})_{t \geq 0},$$

namely, the set of  $\omega$  whose paths behave exactly like a sample path from the set  $B$ , but with the exception that additionally exactly one jump of size in  $(c - \delta_2, c + \delta_2)$  occurs in the interval  $(T_1 \wedge K, (T_1 \wedge K) + \delta]$ . Since  $T_1 \wedge K$  is a finite stopping time, it follows from the strong Markov property of  $\xi$  and from  $P(B) > 0$  that also  $P(C) > 0$ . But for  $\omega \in C$ , with  $\omega' \in B$  and  $\alpha = \alpha(\omega') \in (c - \delta_2, c + \delta_2)$  as in the definition of  $C$ , we obtain

$$\begin{aligned} \int_0^\infty e^{-\xi_s(\omega)} ds &= \int_0^{T_1(\omega')} e^{-\xi_s(\omega')} ds + \int_{T_1(\omega')}^{T_1(\omega') + \delta} e^{-\xi_s(\omega)} ds + e^{-\alpha} \int_{T_1(\omega') + \delta}^\infty e^{-\xi_s(\omega')} ds \\ &\in \left[ z_0 + \int_{T_1(\omega')}^{T_1(\omega') + \delta} e^{-\xi_s(\omega)} ds + e^{-\alpha} \left( x_0 - \varepsilon - z_0 - \int_{T_1(\omega')}^{T_1(\omega') + \delta} e^{-\xi_s(\omega')} ds \right), \right. \\ &\quad \left. z_0 + \int_{T_1(\omega')}^{T_1(\omega') + \delta} e^{-\xi_s(\omega)} ds + e^{-\alpha} (x_0 + \varepsilon - z_0 - \int_{T_1(\omega')}^{T_1(\omega') + \delta} e^{-\xi_s(\omega')} ds) \right] \\ &\subset \left[ z_0 - \delta_1 + e^{-c}(x_0 - z_0 - \varepsilon) + (e^{-c - \delta_2} - e^{-c})(x_0 - z_0 - \varepsilon) - e^{-c + \delta_2} \delta_1, \right. \\ &\quad \left. z_0 + \delta_1 + e^{-c}(x_0 - z_0 + \varepsilon) + (e^{-c + \delta_2} - e^{-c})(x_0 - z_0 + \varepsilon) + e^{-c + \delta_2} \delta_1 \right]. \end{aligned}$$

Since  $y_0 = z_0 + e^{-c}(x_0 - z_0)$ , we see that  $y_0 \in \text{supp } \mathcal{L}(V)$  by choosing  $\varepsilon, \delta_1$  and  $\delta_2$  sufficiently small. So we have shown that  $\text{supp } \mathcal{L}(V)$  is an interval with 0 as its lower endpoint if  $\nu_\xi((0, \infty)) > 0$ .

By a similar reasoning, one can show that  $\text{supp } \mathcal{L}(V)$  is an interval with  $+\infty$  as its upper endpoint if  $\nu_\xi((-\infty, 0)) > 0$ .

It follows that  $\text{supp } \mathcal{L}(V) = [0, \infty)$  if  $\nu_\xi((0, \infty)) > 0$  and  $\nu_\xi((-\infty, 0)) > 0$ . Now suppose that  $\xi$  is of infinite variation with  $\nu_\xi((0, \infty)) > 0$  (but  $\nu_\xi((-\infty, 0)) = 0$ ), or  $\nu_\xi((-\infty, 0)) > 0$

0 (but  $\nu_\xi((0, \infty)) = 0$ ). Then there is  $\alpha > 0$  such that for each  $t_1, t_0 > 0$  with  $t_1 > t_0$  and  $K > 0$  the event

$$\{\xi_s \geq -2, \forall s \in [0, t_0], \quad \xi_s \geq K, \forall s \in [t_0, t_1], \quad \xi_s \geq \alpha s, \forall s \geq t_1\}$$

has a positive probability, since  $\lim_{t \rightarrow \infty} t^{-1}\xi_t$  exists almost surely in  $(0, \infty]$  by [13, Theorems 4.3 and 4.4]) and since  $\text{supp } \mathcal{L}(\xi_t) = \mathbb{R}$  for all  $t > 0$  (cf. [26, Theorem 24.10]). Choosing  $t_0$  small enough and  $t_1, K$  big enough, it follows that  $0 \in \text{supp } \mathcal{L}(V)$  since  $\text{supp } \mathcal{L}(V)$  is closed. On the other hand, since also the event

$$\{\xi_s \leq 2, \forall s \in [0, t_2]\}$$

has positive probability for each  $t_2 > 0$  as a consequence of the infinite variation of  $\xi$ , it follows that  $\text{supp } \mathcal{L}(V)$  is unbounded, hence showing that  $\text{supp } \mathcal{L}(V) = [0, \infty)$  if  $\xi$  is of infinite variation.

Now assume that  $\xi$  is of finite variation with drift  $b \in \mathbb{R}$ ,  $\nu_\xi((0, \infty)) > 0$  and  $\nu_\xi((-\infty, 0)) = 0$ . We already know that  $0 \in \text{supp } \mathcal{L}(V)$ . If  $b \leq 0$ , then the event  $\{\xi_s \leq 2, \forall s \in [0, t_2]\}$  has a positive probability for each  $t_2 > 0$ , and hence  $\text{supp } \mathcal{L}(V)$  is unbounded. If  $b > 0$ , then for each  $\varepsilon > 0$  and  $t_2 > 0$ , the event  $\{\xi_s \leq (b + \varepsilon)s, \forall s \in [0, t_2]\}$  has a positive probability by Shtatland's result (cf. [26, Theorem 43.20]), so that  $\sup \text{supp } \mathcal{L}(V) \geq \int_0^{t_2} e^{-(b+\varepsilon)s} ds$  for each  $t_2 > 0$  and  $\varepsilon > 0$ , and hence  $\sup \text{supp } \mathcal{L}(V) \geq 1/b$ . On the other hand, in that case  $V = \int_0^\infty e^{-\xi_s} ds \leq \int_0^\infty e^{-bs} ds = 1/b$ , so that  $\text{supp } \mathcal{L}(V) = [0, 1/b]$ .

Finally, assume that  $\xi$  is of finite variation with drift  $b > 0$ ,  $\nu_\xi((0, \infty)) = 0$  and  $\nu_\xi((-\infty, 0)) > 0$ . Then  $\text{supp } \mathcal{L}(V)$  is unbounded and by arguments similar to above, using that  $\lim_{t \rightarrow \infty} t^{-1}\xi_t = E[\xi_1] \in (0, b)$ , we see that  $\inf \text{supp } \mathcal{L}(V) = 1/b$ , so that  $\text{supp } \mathcal{L}(V) = [1/b, \infty)$ . This finishes the proof.  $\square$

Now we can characterize the support of  $\mathcal{L}(\int_0^\infty e^{-\xi_s -} d\eta_s)$  when  $\xi$  and  $\eta$  are independent Lévy processes. Observe that Theorem 2.2 below together with Lemma 2.1 provides a complete characterization of all possible cases.

**Theorem 2.2.** *Let  $\xi$  and  $\eta$  be two independent Lévy processes such that  $V := \int_0^\infty e^{-\xi_s -} d\eta_s$  converges almost surely.*

- (i) *Suppose that  $\eta$  is of infinite variation, or that  $\nu_\eta((0, \infty)) > 0$  and  $\nu_\eta((-\infty, 0)) > 0$ . Then  $\text{supp } \mathcal{L}(V) = \mathbb{R}$ .*
- (ii) *Suppose that  $\eta$  is of finite variation with drift  $a$ ,  $\nu_\eta((0, \infty)) > 0$  and  $\nu_\eta((-\infty, 0)) = 0$ . Then for  $a \geq 0$*

$$\text{supp } \mathcal{L}(V) = \begin{cases} [\frac{a}{b}, \infty), & \text{if } \xi \text{ is of finite variation with drift } b > 0 \text{ and } \nu_\xi((0, \infty)) = 0, \\ [0, \infty), & \text{otherwise,} \end{cases}$$

*and for  $a < 0$*

$$\text{supp } \mathcal{L}(V) = \begin{cases} [\frac{a}{b}, \infty), & \text{if } \xi \text{ is a subordinator with drift } b > 0, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

(iii) Suppose that  $\eta$  is of finite variation with drift  $a$ ,  $\nu_\eta((0, \infty)) = 0$  and  $\nu_\eta((-\infty, 0)) > 0$ . Then for  $a > 0$

$$\text{supp } \mathcal{L}(V) = \begin{cases} (-\infty, \frac{a}{b}] , & \text{if } \xi \text{ is a subordinator with drift } b > 0, \\ \mathbb{R}, & \text{otherwise,} \end{cases}$$

and for  $a \leq 0$

$$\text{supp } \mathcal{L}(V) = \begin{cases} (-\infty, \frac{a}{b}] , & \text{if } \xi \text{ is of finite variation with drift } b > 0 \text{ and } \nu_\xi((0, \infty)) = 0, \\ (-\infty, 0], & \text{otherwise.} \end{cases}$$

*Proof.* Denote by  $D([0, \infty), \mathbb{R})$  the set of all real valued càdlàg functions on  $[0, \infty)$ . Since  $\xi$  and  $\eta$  are independent, we can condition on  $\xi = f$  with  $f \in D([0, \infty), \mathbb{R})$  and it follows that, for  $P_\xi$ -almost every  $f \in D([0, \infty), \mathbb{R})$ ,

$$V_f := \int_0^\infty e^{-f(s-)} d\eta_s = \lim_{T \rightarrow \infty} \int_0^T e^{-f(s-)} d\eta_s$$

converges almost surely. Hence we can apply the results in [27] for such  $f$ , and obtain that  $V_f$  is infinitely divisible with Gaussian variance

$$A_f = A_\eta \int_0^\infty e^{-2f(s)} ds$$

and Lévy measure  $\nu_f$ , given by

$$\nu_f(B) = \int_0^\infty ds \int_{\mathbb{R}} \mathbb{1}_B(e^{-f(s)}x) \nu_\eta(dx) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d) \text{ with } 0 \notin B$$

(cf. [27, Theorem 3.10]). In particular,  $A_f > 0$  if and only if  $A_\eta > 0$ ,  $\nu_f((0, \infty)) > 0$  if and only if  $\nu_\eta((0, \infty)) > 0$ , and  $\nu_f((-\infty, 0)) > 0$  if and only if  $\nu_\eta((-\infty, 0)) > 0$ . Further, since  $\lim_{s \rightarrow \infty} f(s) = +\infty$   $P_\xi$ -a.s. ( $f$ ), for any  $\varepsilon > 0$  we conclude that

$$\nu_f((-\varepsilon, \varepsilon) \setminus \{0\}) = \int_0^\infty \nu_\eta((-e^{f(s)}\varepsilon, e^{f(s)}\varepsilon) \setminus \{0\}) ds = \infty$$

provided that  $\nu_\eta \not\equiv 0$ . This shows that  $0 \in \text{supp } \nu_f$ ,  $P_\xi$ -a.s. ( $f$ ). It then follows from [26, Theorem 24.10] that

$$\text{supp } \mathcal{L}(V_f) = \mathbb{R}, \quad P_\xi - \text{a.s.} (f)$$

if  $A_\eta > 0$ , or if  $\nu_\eta((0, \infty)) > 0$  and  $\nu_\eta((-\infty, 0)) > 0$ . Hence in that case  $P(V_f \in B | \xi = f) > 0$   $P_\xi$ -a.s. ( $f$ ) for any open set  $B \neq \emptyset$ , so that  $P(V \in B) = \int P(V_f \in B | \xi = f) dP_\xi(f) > 0$ . Hence  $\text{supp } \mathcal{L}(V) = \mathbb{R}$ , which shows (i).

To show (ii), suppose that  $\eta$  is of finite variation with drift  $a$ , and that  $\nu_\eta((0, \infty)) > 0$  and  $\nu_\eta((-\infty, 0)) = 0$ . Then, for  $P_\xi$ -a.e.  $f$ ,  $V_f \geq a \int_0^\infty e^{-f(s)} ds > -\infty$  and hence  $V_f$  is of finite variation. It then follows from [27, Theorem 3.15] that  $V_f$  has drift  $a \int_0^\infty e^{-f(s)} ds$  and [26, Theorem 24.10] gives

$$\text{supp } \mathcal{L}(V_f) = \left[ a \int_0^\infty e^{-f(s)} ds, \infty \right).$$

Since  $P(V \in B) = \int P(V_f \in B | \xi = f) dP_\xi(f)$ , the assertion (ii) follows from Lemma 2.1. Finally, (iii) follows from (ii) by replacing  $\eta$  by  $-\eta$ .  $\square$

The following result is now immediate.

**Corollary 2.3.** *Let  $\xi$  be a Lévy process drifting to  $+\infty$ , and  $\eta$  another Lévy process, independent of  $\xi$  such that  $\mathcal{L}(\eta_1) \in D_\xi$ . Then  $V = \int_0^\infty e^{-\xi_{s-}} d\eta_s \geq 0$  a.s. if and only if  $\eta$  is a subordinator.*

**Remark 2.4.** (i) Let  $\xi$  and  $\eta$  be two independent Lévy processes such that  $V = \int_0^\infty e^{-\xi_{s-}} d\eta_s$  converges almost surely and consider the associated GOU process  $(V_t)_{t \geq 0}$  defined by (1.2). Then it is easy to see that  $V_n = A_n V_{n-1} + B_n$  for each  $n \in \mathbb{N}$ , where  $((A_n, B_n)^T)_{n \in \mathbb{N}}$  is an i.i.d. sequence of bivariate random vectors given by

$$(A_n, B_n)^T = \left( e^{-(\xi_n - \xi_{n-1})}, e^{-(\xi_n - \xi_{n-1})} \int_{(n-1, n]} e^{\xi_{s-} - \xi_{n-1}} d\eta_s \right)^T$$

(e.g. [22, Lemma 6.2]). Further, if  $V_0$  is chosen to be independent of  $(\xi, \eta)^T$ , then  $(V_0, \dots, V_{n-1})^T$  is independent of  $((A_k, B_k)^T)_{k \geq n}$  for each  $n$ . Since  $\mathcal{L}(V)$  is the stationary marginal distribution of the GOU process, it is also the stationary marginal distribution of the random recurrence equation  $V_n = A_n V_{n-1} + B_n$ ,  $n \in \mathbb{N}$ . We have seen in particular, that the support of  $\mathcal{L}(V)$  was always an interval. Hence it is natural to ask if stationary solutions to arbitrary random recurrence equations will always have an interval as its support. We will see that this is not the case. To be more precise, let  $((A_n, B_n)^T)_{n \in \mathbb{N}}$  be a given i.i.d. sequence of bivariate random vectors. Suppose that  $(X_n)_{n \in \mathbb{N}_0}$  is a strictly stationary sequence which satisfies the random recurrence equation

$$X_n = A_n X_{n-1} + B_n, \quad n \in \mathbb{N}, \quad (2.1)$$

such that  $(X_0, \dots, X_{n-1})$  is independent of  $((A_k, B_k)^T)_{k \geq n}$  (provided that such a solution exists) for every  $n \in \mathbb{N}$ . Then the support of  $\mathcal{L}(X_0)$  does not need to be an interval, even if  $A_n$  is constant and hence  $A_n$  and  $B_n$  are independent. To see this, let  $A_n = 1/3$  and let  $(B_n)_{n \in \mathbb{Z}}$  be an i.i.d. sequence such that  $P(B_n = 0) = P(B_n = 2) = 1/2$ . Then

$$X_n = \sum_{k=0}^{\infty} 3^{-k} B_{n-k}, \quad n \in \mathbb{N}_0, \quad (2.2)$$

defines a stationary solution of (2.1), which is unique in distribution. Obviously, the support of  $\mathcal{L}(X_0)$  is given by the Cantor set

$$\left\{ \sum_{n=0}^{\infty} 3^{-n} z_n : z_n \in \{0, 2\}, \forall n \in \mathbb{N}_0 \right\},$$

which is totally disconnected and not an interval.

(ii) The stationary solution constructed in (2.2) is a  $1/3$ -decomposable distribution (see



[26, Definition 64.1] for the definition). By Proposition 6.2 in [5], there exists a bivariate Lévy process  $(\xi, \eta)^T$  such that  $\xi_t = (\log 3)N_t$  for a Poisson process  $(N_t)_{t \geq 0}$  and such that

$$\int_0^\infty e^{-\xi_{s-}} d\eta_s = \int_0^\infty 3^{-N_{s-}} d\eta_s$$

has the same distribution as  $X_0$  from (2.2). In particular, its support is not an interval. Hence a similar statement to Theorem 2.2 does not hold under dependence.

### 3 Closedness of the range

This section is devoted to show that, as in the well-known case of a deterministic process  $\xi$ , the range  $R_\xi = \Phi_\xi(D_\xi)$  is closed under weak convergence. On the contrary, closedness of  $R_\xi$  under convolution does not hold any more as will be demonstrated in Example 3.6 below.

It will also follow that the inverse mapping  $(\Phi_\xi)^{-1}$  is continuous, provided that  $\Phi_\xi$  is injective. Recall that  $\Phi_\xi$  is injective if, for instance,  $\xi$  is spectrally negative (cf. [5, Theorem 5.3]). Further, for any  $\xi$  drifting to  $+\infty$ ,  $\Phi_\xi$  is always injective when restricted to positive measures  $\mathcal{L}(\eta_1)$  [5, Remark 5.4]. Thus, although  $\Phi_\xi$  need not be continuous (which follows by an argument similar to [5, Example 7.1]), the inverse of  $\Phi_\xi$  restricted to positive measures will turn out to be always continuous.

We start with the following proposition, which shows that the mapping  $\Phi_\xi$  is closed.

**Proposition 3.1.** *Let  $\xi$  be a Lévy process drifting to  $+\infty$ . Then the mapping  $\Phi_\xi$  is closed in the sense that if  $\mathcal{L}(\eta_1^{(n)}) \in D_\xi$ ,  $\eta_1^{(n)} \xrightarrow{d} \eta_1$  and  $\Phi_\xi(\mathcal{L}(\eta_1^{(n)})) \xrightarrow{w} \mu$  for some random variable  $\eta_1$  and probability measure  $\mu$  as  $n \rightarrow \infty$ , then  $\mathcal{L}(\eta_1) \in D_\xi$  and  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$ .*

*Proof.* For  $n \in \mathbb{N}$ , let  $W^{(n)}$  be a random variable such that

$$W^{(n)} \stackrel{d}{=} \int_0^\infty e^{-\xi_{s-}} d\eta_s^{(n)} \quad \text{and } W^{(n)} \text{ is independent of } (\xi, \eta^{(n)})^T,$$

where  $\eta^{(n)}$  is a Lévy process induced by  $\eta_1^{(n)}$  independent of  $\xi$ . Then the limit  $\mathcal{L}(\eta_1)$  is infinitely divisible by [26, Lemma 7.8]) and we can define  $\eta$  as a Lévy process induced by  $\eta_1$ , independent of  $\xi$ . Let  $W$  be a random variable with distribution  $\mu$ , independent of  $(\xi, \eta)^T$ . The proof of [5, Theorem 7.3], more precisely the part leading to Equation (7.12) there, then shows that for every  $t > 0$  we have

$$\left( e^{-\xi_t}, \int_0^t e^{\xi_{s-}} d\eta_s^{(n)} \right)^T \xrightarrow{d} \left( e^{-\xi_t}, \int_0^t e^{\xi_{s-}} d\eta_s \right)^T, \quad n \rightarrow \infty.$$

Due to independence this yields

$$\left( W^{(n)}, e^{-\xi_t}, \int_0^t e^{\xi_{s-}} d\eta_s^{(n)} \right)^T \xrightarrow{d} \left( W, e^{-\xi_t}, \int_0^t e^{\xi_{s-}} d\eta_s \right)^T, \quad n \rightarrow \infty,$$

and since  $\mathcal{L}(W^{(n)})$  is the invariant distribution of the GOU process driven by  $(\xi, \eta^{(n)})^T$ , this implies

$$W^{(n)} \stackrel{d}{=} e^{-\xi_t} \left( W^{(n)} + \int_0^t e^{\xi_{s-}} d\eta_s^{(n)} \right) \xrightarrow{d} e^{-\xi_t} \left( W + \int_0^t e^{\xi_{s-}} d\eta_s \right), \quad n \rightarrow \infty.$$

Since also  $W^{(n)} \xrightarrow{d} W$  as  $n \rightarrow \infty$ , this shows that

$$W \stackrel{d}{=} e^{-\xi_t} \left( W + \int_0^t e^{\xi_{s-}} d\eta_s \right)$$

for any  $t > 0$ . Since the GOU process driven by  $(\xi, \eta)^T$  is a Markov process, this shows that  $\mu = \mathcal{L}(W)$  is the unique invariant distribution of this GOU process. By [4, Theorem 2.1], this shows that  $\int_0^\infty e^{-\xi_{s-}} d\eta_s$  converges a.s., i.e.  $\mathcal{L}(\eta_1) \in D_\xi$ , and that

$$\mu = \mathcal{L}(W) = \mathcal{L} \left( \int_0^\infty e^{-\xi_{s-}} d\eta_s \right) = \Phi_\xi(\mathcal{L}(\eta_1)),$$

giving the claim.  $\square$

In order to show that  $R_\xi$  is closed, we shall first show in Proposition 3.3 below that if a sequence  $(\Phi_\xi(\mathcal{L}(\eta_1^{(n)})))_{n \in \mathbb{N}}$  is tight, then  $(\eta_1^{(n)})_{n \in \mathbb{N}}$  is tight. In order to achieve this, observe first that as a consequence of [19, Lemma 15.15] and Prokhorov's theorem, a sequence  $(\mathcal{L}(\eta_1^{(n)}))_{n \in \mathbb{N}}$  of infinitely divisible distributions on  $\mathbb{R}$  with characteristic triplets  $(\gamma_n, \sigma_n^2, \nu_n)$  is tight if and only if

$$\sup_{n \in \mathbb{N}} \left| \gamma_n + \int_{\mathbb{R}} x \left( \frac{1}{1+x^2} - \mathbf{1}_{|x| \leq 1} \right) \nu_n(dx) \right| < \infty$$

and the sequence  $(\tilde{\nu}_n)_{n \in \mathbb{N}}$  of finite positive measures on  $\mathbb{R}$  with

$$\tilde{\nu}_n(dx) = \sigma_n^2 \delta_0(dx) + \frac{x^2}{1+x^2} \nu_n(dx)$$

is weakly relatively compact (in particular, this implies that  $\sup_{n \in \mathbb{N}} \tilde{\nu}_n(\mathbb{R}) < \infty$ ). Using Prokhorov's theorem for finite measures (e.g. [1, Theorem 7.8.7]), it is easy to see that this is equivalent to

$$\sup_{n \in \mathbb{N}} \sigma_n^2 < \infty, \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \int_{[-1,1]} x^2 \nu_n(dx) < \infty, \tag{3.2}$$

$$\sup_{n \in \mathbb{N}} \nu_n(\mathbb{R} \setminus [-r, r]) < \infty, \quad \forall r > 0, \tag{3.3}$$

$$\lim_{r \rightarrow \infty} \sup_{n \in \mathbb{N}} \nu_n(\mathbb{R} \setminus [-r, r]) = 0, \quad \text{and} \tag{3.4}$$

$$\sup_{n \in \mathbb{N}} |\gamma_n| < \infty. \tag{3.5}$$

The following lemma gives direct uniform estimates for  $\mu([-r, r])$  in terms of the Lévy measure or Gaussian variance of an infinitely divisible distribution  $\mu$  which will be needed to prove Proposition 3.3.

**Lemma 3.2.** Let  $\mu$  be an infinitely divisible distribution on  $\mathbb{R}$  with characteristic triplet  $(\gamma, \sigma^2, \nu)$ . For  $\varepsilon \in (0, 1)$  denote by  $I_\varepsilon$  the set

$$I_\varepsilon := \{z \in \mathbb{R} : 1 - \cos z \geq \varepsilon\}.$$

Then for any  $p \in (0, 1)$  and  $a > 0$ , there is some  $\varepsilon = \varepsilon(a, p) \in (0, 1)$  such that

$$\frac{\lambda^1(I_\varepsilon \cap [-y, y])}{\lambda^1([-y, y])} \geq 1 - p, \quad \forall y \geq a, \quad (3.6)$$

where  $\lambda^1$  denotes the Lebesgue measure on  $\mathbb{R}$ . For  $\delta > 0$ , denote by

$$\|\nu\|_\delta := \nu(\mathbb{R} \setminus [-\delta, \delta])$$

the total mass of  $\nu|_{\mathbb{R} \setminus [-\delta, \delta]}$  and

$$M(\nu) := \int_{[-1, 1]} x^2 \nu(dx).$$

Further, let  $c > 0$  be a constant such that

$$\cos(t) - 1 \leq -ct^2, \quad \forall t \in [-1, 1].$$

Then

$$\mu([-r, r]) \leq 4(e^{-\varepsilon(\delta/r, p)\|\nu\|_\delta}(1 - p) + p), \quad \forall p \in (0, 1), r, \delta > 0, \quad (3.7)$$

$$\mu([-r, r]) \leq 1 - \min\{e^{-\|\nu\|_{2r}}, 1 - e^{-\|\nu\|_{2r}/2}\}, \quad \forall r > 0, \quad (3.8)$$

$$\mu([-r, r]) \leq 2r \int_{-1/r}^{1/r} e^{-M(\nu)ct^2} dt, \quad \forall r \geq 1, \quad (3.9)$$

and

$$\mu([-r, r]) \leq 2r \int_{-1/r}^{1/r} e^{-\sigma^2 t^2/2} dt, \quad \forall r > 0. \quad (3.10)$$

*Proof.* Equation (3.6) is clear. Let  $r > 0$ . Then an application of [19, Lemma 5.1] shows

$$\mu([-r, r]) \leq 2r \int_{-1/r}^{1/r} |\widehat{\mu}(t)| dt = 2r \int_{-1/r}^{1/r} \exp\left(-\sigma^2 t^2/2 + \int_{\mathbb{R}} (\cos(xt) - 1)\nu(dx)\right) dt \quad (3.11)$$

which immediately gives (3.10). Let  $\delta > 0$ . Equation (3.7) is trivial when  $\|\nu\|_\delta = 0$ , and for  $\|\nu\|_\delta > 0$  observe that by (3.11) and Jensen's inequality we can estimate

$$\begin{aligned} \mu([-r, r]) &\leq 2r \int_{-1/r}^{1/r} \exp\left(\int_{|x|>\delta} (\cos(xt) - 1)\|\nu\|_\delta \frac{\nu(dx)}{\|\nu\|_\delta}\right) dt \\ &\leq 2r \int_{-1/r}^{1/r} \left(\int_{|x|>\delta} e^{(\cos(xt)-1)\|\nu\|_\delta} \frac{\nu(dx)}{\|\nu\|_\delta}\right) dt \end{aligned}$$

$$= \int_{|x|>\delta} \left( \frac{2r}{|x|} \int_{-|x|/r}^{|x|/r} e^{(\cos z - 1)\|\nu\|_\delta} dz \right) \frac{\nu(dx)}{\|\nu\|_\delta}. \quad (3.12)$$

By (3.6) we estimate for  $|x| \geq \delta$  and  $p \in (0, 1)$  with  $\varepsilon = \varepsilon(\delta/r, p)$

$$\begin{aligned} & \frac{2r}{|x|} \int_{-|x|/r}^{|x|/r} e^{(\cos z - 1)\|\nu\|_\delta} dz \\ & \leq \frac{4}{\lambda^1([- \frac{|x|}{r}, \frac{|x|}{r}])} \left( e^{-\varepsilon\|\nu\|_\delta} \lambda^1 \left( [- \frac{|x|}{r}, \frac{|x|}{r}] \cap I_\varepsilon \right) + \lambda^1 \left( [- \frac{|x|}{r}, \frac{|x|}{r}] \setminus I_\varepsilon \right) \right) \\ & \leq 4(e^{-\varepsilon\|\nu\|_\delta}(1-p) + p), \end{aligned}$$

which together with (3.12) results in (3.7). Similarly, (3.9) is trivial when  $M(\nu) = 0$ , while for  $M(\nu) > 0$  define the finite measure  $\rho$  on  $[-1, 1]$  by  $\rho(dx) = x^2\nu(dx)$ . We then estimate with (3.11) and Jensen's inequality, for  $r \geq 1$ ,

$$\begin{aligned} \mu([-r, r]) & \leq 2r \int_{-1/r}^{1/r} \exp \left( \int_{[-1, 1]} \frac{\cos(xt) - 1}{x^2} M(\nu) \frac{\rho(dx)}{M(\nu)} \right) dt \\ & \leq 2r \int_{-1/r}^{1/r} \left( \int_{[-1, 1]} \exp \left( \frac{\cos(xt) - 1}{x^2} M(\nu) \right) \frac{\rho(dx)}{M(\nu)} \right) dt \\ & \leq 2r \int_{-1/r}^{1/r} \left( \int_{[-1, 1]} e^{-ct^2 M(\nu)} \frac{\rho(dx)}{M(\nu)} \right) dt, \end{aligned}$$

which gives (3.9). Finally, let us prove Equation (3.8). This is again trivial when  $\|\nu\|_{2r} = 0$ , so assume  $\|\nu\|_{2r} > 0$ . By symmetry, we can assume without loss of generality that

$$\nu((-\infty, -2r)) \geq \|\nu\|_{2r}/2 > 0.$$

Let  $(X_t)_{t \geq 0}$  be a Lévy process with  $\mathcal{L}(X_1) = \mu$ , and define

$$Y_t := \sum_{0 < s \leq t, \Delta X_s < -2r} \Delta X_s \quad \text{and} \quad Z_t := X_t - Y_t, \quad t \in \mathbb{R},$$

where  $\Delta X_s := X_s - X_{s-}$  denotes the jump size of  $X$  at time  $s$ . Then  $(Y_t)_{t \geq 0}$  and  $(Z_t)_{t \geq 0}$  are two independent Lévy processes, and  $(Y_t)_{t \geq 0}$  is a compound Poisson process with Lévy measure  $\nu|_{(-\infty, -2r)}$ . Denote by  $(N_t)_{t \geq 0}$  the underlying Poisson process in  $(Y_t)_{t \geq 0}$  which counts the number of jumps of  $(Y_t)_{t \geq 0}$ . Then

$$\begin{aligned} \mu(\mathbb{R} \setminus [-r, r]) & = P(|Y_1 + Z_1| > r) \\ & \geq P(|Z_1| > r, Y_1 = 0) + P(|Z_1| \leq r, Y_1 < -2r) \\ & = P(|Z_1| > r) P(N_1 = 0) + P(|Z_1| \leq r) P(N_1 \geq 1) \\ & = P(|Z_1| > r) e^{-\nu((-\infty, -2r))} + (1 - P(|Z_1| > r))(1 - e^{-\nu((-\infty, -2r))}) \\ & \geq \min\{e^{-\nu((-\infty, -2r))}, 1 - e^{-\nu((-\infty, -2r))}\} \\ & \geq \min\{e^{-\|\nu\|_{2r}}, 1 - e^{-\|\nu\|_{2r}/2}\}, \end{aligned}$$

which implies (3.8). □

The next result is the key step in proving closedness of  $R_\xi$ .

**Proposition 3.3.** *Let  $\xi$  be a Lévy process drifting to  $+\infty$  and  $(\mathcal{L}(\eta_1^{(n)}))_{n \in \mathbb{N}}$  be a sequence in  $D_\xi$  such that  $(\mu_n := \Phi_\xi(\mathcal{L}(\eta_1^{(n)})))_{n \in \mathbb{N}}$  is tight. Then also  $(\eta_1^{(n)})_{n \in \mathbb{N}}$  is tight.*

*Proof.* Denote by  $(\gamma_n, \sigma_n^2, \nu_n)$  the characteristic triplet of  $\eta_1^{(n)}$ . We have to show that conditions (3.1) – (3.5) are satisfied. Since  $\xi$  has càdlàg paths, there are some  $0 < D_1 \leq 1 \leq D_2 < \infty$  and some measurable sets  $A_n \subset D([0, \infty), \mathbb{R})$  with

$$D_1 \leq e^{-f(s)} \leq D_2 \quad \forall f \in A_n, s \in [0, 1],$$

$$\int_0^\infty e^{-f(s-)} d\eta_s^{(n)} \text{ converges a.s., } \forall f \in A_n,$$

and  $P((\xi_s)_{s \in [0, 1]} \in A_n) \geq 1/2$ . Conditioning on  $\xi = f$  for  $f \in D([0, \infty), \mathbb{R})$ , we obtain for any  $r > 0$  by independence of  $\xi$  and  $\eta^{(n)}$

$$\begin{aligned} \mu_n(\mathbb{R} \setminus [-r, r]) &\geq \int_{A_n} P\left(\left|\int_0^\infty e^{-f(s-)} d\eta_s^{(n)}\right| > r\right) P_\xi(df) \\ &\geq \frac{1}{2} \left(1 - \sup_{f \in A_n} P\left(\left|\int_0^\infty e^{-f(s-)} d\eta_s^{(n)}\right| \leq r\right)\right). \end{aligned} \quad (3.13)$$

For fixed  $f \in A_n$  the distribution of  $\int_0^\infty e^{-f(s-)} d\eta_s^{(n)}$  is infinitely divisible with Gaussian variance

$$\sigma_{f,n}^2 = \sigma_n^2 \int_0^\infty (e^{-f(s)})^2 ds \geq D_1^2 \sigma_n^2 \quad (3.14)$$

and Lévy measure  $\nu_{f,n}$  satisfying

$$\nu_{f,n}(B) = \int_0^\infty ds \int_{\mathbb{R}} \mathbf{1}_B(e^{-f(s)}x) \nu_n(dx) \quad (3.15)$$

for any Borel set  $B \subset \mathbb{R} \setminus \{0\}$  (cf. [27, Theorem 3.10]). In particular, for  $f \in A_n$  and any  $\delta > 0$ ,

$$\nu_{f,n}(\mathbb{R} \setminus [-\delta, \delta]) \geq \int_0^1 ds \int_{\mathbb{R}} \mathbf{1}_{\mathbb{R} \setminus [-\delta e^{f(s)}, \delta e^{f(s)}]}(x) \nu_n(dx) \geq \nu_n(\mathbb{R} \setminus [-\delta/D_1, \delta/D_1]). \quad (3.16)$$

From (3.15) we obtain

$$\int_{[-1, 1]} t^2 \nu_{f,n}(dt) = \int_0^\infty ds \int_{\mathbb{R}} (e^{-f(s)}x)^2 \mathbf{1}_{\{|e^{-f(s)}x| \leq 1\}}(x) \nu_n(dx),$$

for  $f \in A_n$ , hence

$$\int_{[-1, 1]} t^2 \nu_{f,n}(dt) \geq D_1^2 \int_{[-1, 1]} x^2 \mathbf{1}_{\{|D_2 x| \leq 1\}}(x) \nu_n(dx) = D_1^2 \int_{[-1/D_2, 1/D_2]} x^2 \nu_n(dx). \quad (3.17)$$

Now suppose (3.1) were violated. Then by (3.13), (3.10) and (3.14) we conclude that

$$\sup_{n \in \mathbb{N}} \{\mu_n(\mathbb{R} \setminus [-r, r])\} \geq \frac{1}{2} \sup_{n \in \mathbb{N}} \left\{ 1 - 2r \int_{-1/r}^{1/r} e^{-D_1^2 \sigma_n^2 t^2 / 2} dt \right\} = \frac{1}{2}$$

for every  $r > 0$ , contradicting tightness of  $(\mu_n)_{n \in \mathbb{N}}$ . Hence (3.1) must be true.

Now suppose that (3.3) were violated, so that there is some  $\delta > 0$  such that  $\sup_{n \in \mathbb{N}} \|\nu_n\|_\delta = \infty$  with the notions of Lemma 3.2. Let  $p \in (0, 1/4)$  be arbitrary. Then by (3.7) and (3.16), we have for every  $f \in A_n$ , with  $\epsilon = \epsilon(D_1 \delta / r, p)$  as defined in Lemma 3.2, that

$$\begin{aligned} P \left( \left| \int_0^\infty e^{-f(s-)} d\eta_s^{(n)} \right| \leq r \right) &\leq 4 \left( e^{-\epsilon(D_1 \delta / r, p) \|\nu_{f,n}\|_{D_1 \delta}} (1-p) + p \right) \\ &\leq 4e^{-\epsilon(D_1 \delta / r, p) \|\nu_n\|_\delta} (1-p) + 4p. \end{aligned} \quad (3.18)$$

From (3.13) we then obtain that

$$\sup_{n \in \mathbb{N}} \{\mu_n(\mathbb{R} \setminus [-r, r])\} \geq \frac{1}{2} (1 - 4p) > 0, \quad \forall r > 0,$$

which again contradicts tightness of  $(\mu_n)_{n \in \mathbb{N}}$  so that (3.3) must hold.

Now suppose that (3.4) were violated. Then there is some  $a > 0$  and a sequence  $(\delta_k)_{k \in \mathbb{N}}$  of positive real numbers tending to  $+\infty$  and an index  $n(k) \in \mathbb{N}$  for each  $k$  such that

$$\|\nu_{n(k)}\|_{2\delta_k/D_1} \geq a, \quad \forall k \in \mathbb{N}.$$

Let  $p \in (0, 1/4)$  be arbitrary and choose  $\epsilon = \epsilon(D_1, p)$  as in Lemma 3.2. Let  $b > 0$  be such that

$$b_1 := 4 \left( e^{-\epsilon(D_1, p)b} (1-p) + p \right) < 1.$$

Let  $f \in A_n$ . Then if  $\|\nu_{f,n(k)}\|_{D_1 \delta_k} \geq b$  we have

$$P \left( \left| \int_0^\infty e^{-f(s-)} d\eta_s^{(n(k))} \right| \leq \delta_k \right) \leq b_1 < 1$$

by (3.18), while if  $\|\nu_{f,n(k)}\|_{D_1 \delta_k} < b$  we obtain from (3.8) and (3.16) that

$$\begin{aligned} P \left( \left| \int_0^\infty e^{-f(s-)} d\eta_s^{(n(k))} \right| \leq \delta_k \right) &\leq 1 - \min\{e^{-\|\nu_{f,n(k)}\|_{2\delta_k}}, 1 - e^{-\|\nu_{f,n(k)}\|_{2\delta_k}/2}\} \\ &\leq 1 - \min\{e^{-b}, 1 - e^{-\|\nu_{n(k)}\|_{2\delta_k/D_1}/2}\} \\ &\leq 1 - \min\{e^{-b}, 1 - e^{-a/2}\}. \end{aligned}$$

From (3.13) we then conclude

$$\mu_{n(k)}(\mathbb{R} \setminus [-\delta_k, \delta_k]) \geq \frac{1}{2} (1 - \max\{b_1, 1 - e^{-b}, e^{-a/2}\}) > 0 \quad \forall k \in \mathbb{N}.$$

In particular,

$$\limsup_{r \rightarrow \infty} \sup_{n \in \mathbb{N}} \{\mu_n(\mathbb{R} \setminus [-r, r])\} \geq \frac{1}{2} (1 - \max\{b_1, 1 - e^{-b}, e^{-a/2}\}) > 0,$$

which again contradicts tightness of  $(\mu_n)_{n \in \mathbb{N}}$ . We conclude that also (3.4) must be valid.

Now suppose that (3.2) were violated, but (3.3) holds. Then by (3.13), (3.9), (3.17) and with  $c$  from Lemma 3.2 we have for every  $r \geq 1$

$$\sup_{n \in \mathbb{N}} \{ \mu_n(\mathbb{R} \setminus [-r, r]) \} \geq \frac{1}{2} \sup_{n \in \mathbb{N}} \left\{ 1 - 2r \int_{-1/r}^{1/r} \exp \left( -D_1^2 c t^2 \int_{[-1/D_2, 1/D_2]} x^2 \nu_n(dx) \right) dt \right\} = \frac{1}{2},$$

where we have used that (3.3) together with  $\sup_{n \in \mathbb{N}} \int_{[-1, 1]} x^2 \nu_n(dx) = \infty$  imply  $\sup_{n \in \mathbb{N}} \int_{[-1/D_2, 1/D_2]} x^2 \nu_n(dx) = \infty$ . This again contradicts tightness of  $(\mu_n)_{n \in \mathbb{N}}$  so that (3.2) must hold.

Finally, suppose that (3.5) were violated but that (3.1)–(3.4) hold. Then there is a subsequence of  $(\gamma_n)_{n \in \mathbb{N}}$  which diverges to  $+\infty$  or  $-\infty$ , and without loss of generality assume that this is  $(\gamma_n)_{n \in \mathbb{N}}$ . Since  $(\mu_n)_{n \in \mathbb{N}}$  is tight by assumption, there is a subsequence of  $(\mu_n)_{n \in \mathbb{N}}$  which converges weakly, and for the convenience of notation assume again that  $(\mu_n)_{n \in \mathbb{N}}$  converges weakly to some distribution  $\mu$ . Let the Lévy process  $U$  with characteristic triplet  $(\gamma_U, \sigma_U^2, \nu_U)$  be related to  $\xi$  by  $\mathcal{E}(U)_t = e^{-\xi t}$ , where  $\mathcal{E}(U)$  denotes the stochastic exponential of  $U$ . Then it follows from [5, Corollary 3.2 and Equation (4.1)] that

$$\begin{aligned} \gamma_n \int_{\mathbb{R}} f'(x) \mu_n(dx) &= -\frac{1}{2} \sigma_n^2 \int_{\mathbb{R}} f''(x) \mu_n(dx) - \int_{\mathbb{R}} \mu_n(dx) \int_{\mathbb{R}} (f(x+y) - f(x) - f'(x)y \mathbf{1}_{|y| \leq 1}) \nu_n(dy) \\ &\quad - \gamma_U \int_{\mathbb{R}} f'(x)x \mu_n(dx) - \frac{1}{2} \sigma_U^2 \int_{\mathbb{R}} f''(x)x^2 \mu_n(dx) \\ &\quad - \int_{\mathbb{R}} \mu_n(dx) \int_{\mathbb{R}} (f(x+xy) - f(x) - f'(x)xy \mathbf{1}_{|y| \leq 1}) \nu_U(dy) \end{aligned}$$

for every function  $f \in C_c^2(\mathbb{R})$ . Consider the right hand side of this equation. The first summand remains bounded in  $n$  by (3.1) and weak convergence of  $\mu_n$ , and the second remains bounded in  $n$  by (3.2) and (3.3), since

$$|f(x+y) - f(x) - f'(x)y \mathbf{1}_{|y| \leq 1}| \leq 2\|f\|_{\infty} \mathbf{1}_{|y| > 1} + \|f''\|_{\infty} y^2 \mathbf{1}_{|y| \leq 1}$$

(cf. [5, Proof of Lemma 4.2]), where  $\|\cdot\|_{\infty}$  denotes the supremum norm. The third and fourth summands converge by weak convergence of  $\mu_n$ , and the fifth summand remains bounded in  $n$  by [5, Equation (3.6)] (actually, the fifth summand can be shown to converge). We conclude also that  $\gamma_n \int_{\mathbb{R}} f'(x) \mu_n(dx)$  must be bounded in  $n$  for every  $f \in C_c^2(\mathbb{R})$ . Choosing  $f \in C_c^2(\mathbb{R})$  such that  $\int_{\mathbb{R}} f'(x) \mu(dx) \neq 0$ , we obtain that  $(\gamma_n)_{n \in \mathbb{N}}$  must be bounded and hence the desired contradiction. Summing up, we have verified (3.1) – (3.5) so that  $(\eta_1^{(n)})_{n \in \mathbb{N}}$  must be tight.  $\square$

Now define

$$\begin{aligned} D_{\xi}^+ &:= \{ \mathcal{L}(\eta_1) \in D_{\xi} : \eta_1 \geq 0 \text{ a.s.} \}, \\ \Phi_{\xi}^+ &:= (\Phi_{\xi})|_{D_{\xi}^+}, \end{aligned}$$

and

$$R_\xi^+ := \Phi_\xi(D_\xi^+) = \Phi_\xi^+(D_\xi^+).$$

By Corollary 2.3,

$$R_\xi^+ = R_\xi \cap \{\mu \in \mathcal{P}(\mathbb{R}) : \text{supp } \mu \subset [0, \infty)\}.$$

We now show closedness of  $R_\xi$  under weak convergence and that the inverse of  $\Phi_\xi$  (provided that it exists) is continuous.

**Theorem 3.4.** *Let  $\xi = (\xi_t)_{t \geq 0}$  be a Lévy process drifting to  $+\infty$ .*

- (i) *Then  $R_\xi$  and  $R_\xi^+$  are closed under weak convergence.*
- (ii) *If  $\Phi_\xi$  is injective, then the inverse  $\Phi_\xi^{-1} : R_\xi \rightarrow D_\xi$  is continuous with respect to the topology induced by weak convergence.*
- (iii) *The inverse  $(\Phi_\xi^+)^{-1} : R_\xi^+ \rightarrow D_\xi^+$  is continuous.*

*Proof.* (i) Let  $(\mu_n = \Phi_\xi(\mathcal{L}(\eta_1^{(n)})))_{n \in \mathbb{N}}$  be a sequence in  $R_\xi$  which converges weakly to some  $\mu \in \mathcal{P}(\mathbb{R})$ . Then  $(\mu_n)_{n \in \mathbb{N}}$  is tight, and by Proposition 3.3,  $(\eta_1^{(n)})_{n \in \mathbb{N}}$  must be tight, too. Hence there is a subsequence  $(\eta_1^{(n_k)})_{k \in \mathbb{N}}$  which converges weakly to some random variable  $\eta_1$ . It then follows from Proposition 3.1 that also  $\mathcal{L}(\eta_1) \in D_\xi$  and that  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$ . Hence  $\mu \in R_\xi$  so that  $R_\xi$  is closed. Since  $\{\mu \in \mathcal{P}(\mathbb{R}) : \text{supp } \mu \subset [0, \infty)\}$  is closed, this gives also closedness of  $R_\xi^+$ .

(ii) Let  $(\mu_n = \Phi_\xi(\mathcal{L}(\eta_1^{(n)})))_{n \in \mathbb{N}}$  be a sequence in  $R_\xi$  which converges weakly to some  $\mu$ . By Proposition 3.3,  $(\eta_1^{(n)})_{n \in \mathbb{N}}$  is tight. Let  $(\eta_1^{(k_n)})_{k \in \mathbb{N}}$  be a subsequence which converges weakly to some  $\eta_1$ , say. Then  $\mathcal{L}(\eta_1) \in D_\xi$  and  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$  by Proposition 3.1, and since  $\Phi_\xi$  is injective we have  $\mathcal{L}(\eta_1) = \Phi_\xi^{-1}(\mu)$ . Since the convergent subsequence was arbitrary, this shows that  $\mathcal{L}(\eta_1^{(n)}) = \Phi_\xi^{-1}(\mu_n)$  converges weakly to  $\Phi_\xi^{-1}(\mu)$  as  $n \rightarrow \infty$  (cf. [10, Corollary to Theorem 25.10]). Hence  $\Phi_\xi$  is continuous.

(iii) This can be proved in complete analogy to (ii). □

**Remark 3.5.** Closedness of  $R_\xi^+$  under weak convergence and continuity of  $(\Phi_\xi^+)^{-1}$  can also be proved in a simpler way by circumventing Proposition 3.3 but using a formula for the Laplace transforms of  $\eta_1^{(n)}$  and  $\mu_n$  (cf. [5, Remark 4.5], or Theorem 4.1 below), and showing that  $\mu^{(n)} \xrightarrow{w} \mu$  implies convergence of the Laplace transforms of  $\eta_1^{(n)}$ . A similar approach for showing closedness of  $R_\xi$  is not evident since there is not a similarly convenient formula for the Fourier transforms available, but only one in terms of suitable two-sided limits (cf. [5, Equation (4.7)]).

As a consequence of Theorem 3.4, we can now show that  $R_\xi$  will not be closed under convolution if  $\xi$  is non-deterministic and satisfies a suitable moment condition. We conjecture that  $R_\xi$  will never be closed under convolution unless  $\xi$  is deterministic.

**Corollary 3.6.** *Let  $\xi = (\xi_t)_{t \geq 0}$  be a non-deterministic Lévy process drifting to  $+\infty$  such that  $E[e^{-2\xi_1}] < 1$ . Then  $R_\xi$  is not closed under convolution.*



*Proof.* Let  $(\eta_t)_{t \geq 0}$  be a symmetric compound Poisson process with Lévy measure  $\nu = \delta_{-1} + \delta_1$ , where  $\delta_a$  denotes the Dirac measure at  $a$ . Then  $\mathcal{L}(\eta_1) \in D_\xi$  and  $V := \int_0^\infty e^{-\xi s} d\eta_s$  is symmetric, too, and since by [3, Theorem 3.3] we have  $E[V^2] < \infty$ , this yields  $E[V] = 0$ . Now let  $(V_i)_{i \in \mathbb{N}}$  be an i.i.d. family of independent copies of  $V$ . Then by the Central Limit Theorem,

$$\mathcal{L}\left(n^{-\frac{1}{2}}(V_1 + \dots + V_n)\right) \rightarrow \mathcal{N}(0, \text{Var}(V)), \quad n \rightarrow \infty,$$

with  $\text{Var}(V) \neq 0$ . If  $R_\xi$  was closed under convolution, we had  $\mathcal{L}(n^{-\frac{1}{2}}(V_1 + \dots + V_n)) \in R_\xi$  and due to closedness of  $R_\xi$  under weak convergence this gave  $\mathcal{N}(0, \text{Var}(V)) \in R_\xi$ . This contradicts [5, Theorem 6.4].  $\square$

## 4 A general criterion for a positive distribution to be in the range

From this section on, we restrict ourselves to positive distributions in  $R_\xi$ , i.e. we only consider  $\Phi_\xi^+$  and  $R_\xi^+$  as defined in Section 3. We start by giving a general criterion to decide whether a positive distribution is in the range  $R_\xi^+$  of  $\Phi_\xi^+$  for a given Lévy process  $\xi$ .

**Theorem 4.1.** *Let  $\xi$  be a Lévy process drifting to  $+\infty$ . Let  $\mu = \mathcal{L}(V)$  be a probability measure on  $[0, \infty)$  with Laplace exponent  $\psi_V$ . Then  $\mu \in R_\xi^+$  if and only if the function*

$$\begin{aligned} g_\mu : (0, \infty) &\rightarrow \mathbb{R} \\ g_\mu(u) &:= \left(\gamma_\xi - \frac{\sigma_\xi^2}{2}\right)u\psi'_V(u) - \frac{\sigma_\xi^2}{2}u^2 \left(\psi''_V(u) + (\psi'_V(u))^2\right) \\ &\quad - \int_{\mathbb{R}} \left(e^{\psi_V(ue^{-y}) - \psi_V(u)} - 1 + u\psi'_V(u)y\mathbf{1}_{|y| \leq 1}\right) \nu_\xi(dy), \quad u > 0, \end{aligned} \quad (4.1)$$

*defines the Laplace exponent of some subordinator  $\eta$ , i.e. if there is some subordinator  $\eta$  such that*

$$E[e^{-\eta u}] = e^{g_\mu(u)}, \quad \forall u > 0. \quad (4.2)$$

*In that case,  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$ .*

Using a Taylor expansion for  $|y| \leq 1$ , it is easy to see that the integral defining  $g_\mu$  converges for every distribution  $\mu$  on  $[0, \infty)$ .

*Proof.* Observe first that

$$-E[Ve^{-uV}] = \mathbb{L}'_V(u) = \psi'_V(u)e^{\psi_V(u)} \quad (4.3)$$

$$E[V^2e^{-uV}] = \mathbb{L}''_V(u) = \psi''_V(u)e^{\psi_V(u)} + (\psi'_V(u))^2e^{\psi_V(u)} \quad (4.4)$$

for  $u > 0$ . Hence

$$g_\mu(u)\mathbb{L}_V(u)$$

$$\begin{aligned}
&= -u\gamma_\xi E[Ve^{-uV}] - \frac{\sigma_\xi^2}{2} (E[V^2e^{-uV}]u^2 - E[Ve^{-uV}]u) \\
&\quad - \int_{(-1,\infty)} (\mathbb{L}_V(ue^{-y}) - \mathbb{L}_V(u) - uE[Ve^{-uV}]y\mathbb{1}_{|y|\leq 1}) \nu_\xi(dy), \quad \forall u > 0. \quad (4.5)
\end{aligned}$$

Now if  $\mu = \mathcal{L}(V) \in R_\xi^+$ , let  $\mathcal{L}(\eta_1) \in D_\xi^+$  such that  $\mu = \mathcal{L}(V) = \Phi_\xi(\mathcal{L}(\eta_1))$ . Then  $g_\mu = \log \mathbb{L}_\eta$  by Remark 4.5 of [5], so that (4.2) is satisfied.

Conversely, suppose that  $V \geq 0$ , and let  $\eta$  be a subordinator such that (4.2) is true. Define the Lévy process  $U$  by  $e^{-\xi t} = \mathcal{E}(U)_t$ , where  $U$  denotes the stochastic exponential of  $U$ . Then by [5, Remark 4.5] and (4.5), (4.2) is equivalent to

$$\begin{aligned}
&\log \mathbb{L}_\eta(u) \mathbb{L}_V(u) \\
&= u\gamma_U E[Ve^{-uV}] - \frac{\sigma_U^2 u^2}{2} E[V^2e^{-uV}] \\
&\quad - \int_{(-1,\infty)} (\mathbb{L}_V(u(1+y)) - \mathbb{L}_V(u) + uE[Ve^{-uV}]y\mathbb{1}_{|y|\leq 1}) \nu_U(dy), \quad \forall u > 0,
\end{aligned}$$

and a direct computation using (4.3) and (4.4) shows that this in turn is equivalent to

$$\begin{aligned}
0 &= \int_{[0,\infty)} \left( f'(x)(x\gamma_U + \gamma_\eta^0) + \frac{1}{2}f''(x)x^2\sigma_U^2 \right) \mu(dx) \\
&\quad + \int_{[0,\infty)} \mu(dx) \int_{(-1,\infty)} (f(x+xy) - f(x) - f'(x)xy\mathbb{1}_{|y|\leq 1}) \nu_U(dy) \\
&\quad + \int_{[0,\infty)} \mu(dx) \int_{[0,\infty)} (f(x+y) - f(x)) \nu_\eta(dy) \quad (4.6)
\end{aligned}$$

for all functions  $f \in \mathcal{G} := \{h \in C_b^2(\mathbb{R}) : \exists u > 0 \text{ such that } h(x) = e^{-ux}, \forall x \geq 0\}$ , where  $\gamma_\eta^0$  denotes the drift of  $\eta$ . Observe that (4.6) is also trivially true for  $f \equiv 1$ . Denote by

$$\begin{aligned}
\mathbf{R}[\mathcal{G}] &:= \{h \in C_b^2(\mathbb{R}) : \exists n \in \mathbb{N}_0, \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}, \exists u_1, \dots, u_n \geq 0 \\
&\quad \text{such that } h(x) = \sum_{k=1}^n \lambda_k e^{-u_k x} \forall x \geq 0\}
\end{aligned}$$

the algebra generated by  $\mathcal{G}$ . By linearity, (4.6) holds true also for all  $f \in \mathbf{R}[\mathcal{G}]$ . Since  $\mathcal{G}$  is strongly separating and since for each  $x \in \mathbb{R}$  there exists  $h \in \mathcal{G}$  such that  $g'(x) \neq 0$ , the set  $\mathcal{G}$  satisfies condition (N) of [20, Definition 1.4.1], and hence  $\mathbf{R}[\mathcal{G}]$  is dense in  $\mathcal{S}^2(\mathbb{R})$  by [20, Corollary 1.4.10], where

$$\mathcal{S}^2(\mathbb{R}) := \{h \in C^2(\mathbb{R}) : \lim_{|x| \rightarrow \infty} (1 + |x|^2)^k (|h(x)| + |h'(x)| + |h''(x)|) = 0, \forall k \in \mathbb{N}_0\}$$

is the space of rapidly decreasing functions of order 2, endowed with the usual topology (cf. [20, Definition 0.1.8]). In particular, for every  $f \in C_c^2(\mathbb{R}) \subset \mathcal{S}^2(\mathbb{R})$  there exists a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathbf{R}[\mathcal{G}]$  such that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} [(1 + |x|^2) (|f_n(x) - f(x)| + |f'_n(x) - f'(x)| + |f''_n(x) - f''(x)|)] = 0.$$

Since (4.6) holds for each  $f_n$ , an application of Lebesgue's dominated convergence theorem shows that (4.6) also holds for  $f \in C_c^2(\mathbb{R})$ ; remark that Lebesgue's theorem can be applied by Equations (3.5) and (3.6) in [5] for the integral with respect to  $\nu_U$  and  $\mu$  and by observing that

$$|f(x+y) - f(x)| \leq 2\|f\|_\infty \mathbf{1}_{y>1} + \|f'\|_\infty y \mathbf{1}_{0<y\leq 1}$$

for the integral with respect to  $\nu_\eta$  and  $\mu$ .

Since  $C_c^2(\mathbb{R})$  is a core for the Feller process

$$W_t^x = x + \int_0^t W_{s-}^x dU_s + \eta_t \quad (4.7)$$

with generator

$$\begin{aligned} A^W f(x) &= f'(x)(x\gamma_U + \gamma_\eta^0) + \frac{1}{2}f''(x)x^2\sigma_U^2 \\ &\quad + \int_{(-1,\infty)} (f(x+xy) - f(x) - f'(x)xy\mathbf{1}_{|y|\leq 1})\nu_U(dy) \\ &\quad + \int_{[0,\infty)} (f(x+y) - f(x))\nu_\eta(dy) \end{aligned}$$

for  $f \in C_c^2(\mathbb{R})$  (cf. [5, Theorem 3.1 and Corollary 3.2] and [26, Equation (8.6)]), we have that  $\int_{\mathbb{R}} A^W f(x) \mu(dx) = 0$  for all  $f$  from a core, and hence  $\mu = \mathcal{L}(V)$  is an invariant measure for the GOU process (4.7) by [21, Theorem 3.37]. By [4, Theorem 2.1(a)], this implies that  $\int_0^\infty e^{-\xi s-} d\eta_s$  converges a.s. and that  $\mathcal{L}(\int_0^\infty e^{-\xi s-} d\eta_s) = \mu$ , so that  $\mathcal{L}(\eta_1) \in D_\xi^+$  and  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$ , completing the proof.  $\square$

**Remark 4.2.** To obtain a similar handy criteria for a non-positive distribution to be in the range  $D_\xi$  seems harder. A general *necessary* condition in this vein for a distribution  $\mu = \mathcal{L}(V)$  to be in the range  $R_\xi$ , where  $\xi$  is a Lévy process with characteristic triplet  $(\gamma_\xi, \sigma_\xi^2, \nu_\xi)$ , can be derived from Equation (4.7) in [5]. If further  $E[V^2] < \infty$ , and  $\log \phi_\eta(u)$  denotes the characteristic exponent of a Lévy process  $\eta$  such that  $E[e^{iu\eta_1}] = \phi_\eta(u)$ ,  $u \in \mathbb{R}$ , then by Equation (4.8) in [5],

$$\begin{aligned} \phi_V(u) \log \phi_\eta(u) &= \gamma_\xi u \phi_V'(u) - \frac{\sigma_\xi^2}{2} (u^2 \phi_V''(u) + u \phi_V'(u)) \\ &\quad - \int_{\mathbb{R}} (\phi_V(ue^{-y}) - \phi_V(u) + uy \phi_V'(u) \mathbf{1}_{|y|\leq 1}) \nu_\xi(du). \end{aligned} \quad (4.8)$$

In [6, Example 3.2], this equation has been derived using the theory of symbols. Hence, a necessary condition for  $V$  with  $E[V^2] < \infty$  to be in  $R_\xi$  is that there is a Lévy process  $\eta$ , such that the right-hand side of (4.8) can be expressed as  $\phi_V(u) \log \phi_\eta(u)$ ,  $u \in \mathbb{R}$ . In Example 4.3 of [6] it has been shown that the existence of some Lévy process  $\eta$  such that the right-hand side of (4.8) can be expressed as  $\phi_V(u) \log \phi_\eta(u)$  is also *sufficient* for  $\mu = \mathcal{L}(V)$  with  $E[V^2] < \infty$  to be in  $R_\xi$ , hence this is a *necessary and sufficient* condition for  $\mathcal{L}(V)$  with  $E[V^2] < \infty$  to be in  $R_\xi$ , similar to Theorem 4.1. Without the assumption  $EV^2 < \infty$ , a necessary and sufficient condition is not established at the moment.

We conclude this section with the following results:

**Lemma 4.3.** *Let  $\xi$  be a spectrally negative Lévy process of infinite variation, drifting to  $+\infty$ . Then every element in  $R_\xi^+$  is selfdecomposable and of finite variation with drift 0.*

*Proof.* That any element in  $R_\xi^+$  must be selfdecomposable has been shown in [7], since  $\xi$  is spectrally negative. Since every element in  $R_\xi^+$  is positive, it must be of finite variation, and it follows from Theorem 2.2 and [26, Theorem 24.10] that the drift must be 0.  $\square$

**Remark 4.4.** It is well known that a selfdecomposable distribution cannot have finite non-zero Lévy measure, in particular it cannot be a compound Poisson distribution, which follows for instance immediately from [26, Corollary 15.11]. This applies in particular to exponential functionals of Lévy processes with spectrally negative  $\xi$ . However, even if  $\xi$  is not spectrally negative, and  $(\xi, \eta)^T$  is a bivariate (possibly dependent) Lévy process, then  $\int_0^\infty e^{-\xi_{s-}} d\eta_s$  (provided it converges) still cannot be a non-trivial compound Poisson distribution, with or without drift. For if  $c$  denotes the drift of a non-trivial compound Poisson distribution with drift, then this distribution must have an atom at  $c$ . However, e.g. by [7, Theorem 2.2],  $\mathcal{L}(\int_0^\infty e^{-\xi_{s-}} d\eta_s)$  must be continuous unless constant. In other words, if  $\int_0^\infty e^{-\xi_{s-}} d\eta_s$  is infinitely divisible, non-constant and has no Gaussian part, then its Lévy measure must be infinite. In particular, it follows that if  $\eta$  is a subordinator and  $\int_0^\infty e^{-\xi_{s-}} d\eta_s$  is infinitely divisible and non-constant, then its Lévy measure must be infinite.

## 5 Some results on $R_\xi^+$ when $\xi$ is a Brownian motion

If  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ , with  $\sigma, a > 0$  and  $(B_t)_{t \geq 0}$  is a standard Brownian motion, then by Lemma 4.3,  $R_\xi^+$  is a subset of  $L(\mathbb{R}_+)$ , the class of selfdecomposable distributions on  $\mathbb{R}_+$ . Recall that a distribution  $\mu = \mathcal{L}(V)$  on  $\mathbb{R}_+$  is selfdecomposable if and only if it is infinitely divisible with non-negative drift and its Lévy measure has a Lévy density of the form  $(0, \infty) \rightarrow [0, \infty)$ ,  $x \mapsto x^{-1}k(x)$  with a non-increasing function  $k = k_V : (0, \infty) \rightarrow [0, \infty)$  (cf. [26, Corollary 15.11]). Further, to every distribution  $\mu = \mathcal{L}(V) \in L(\mathbb{R}_+)$  there exists a subordinator  $X = (X_t)_{t \geq 0} = (X_t(\mu))_{t \geq 0}$ , unique in distribution, such that

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right), \quad (5.1)$$

(cf. [18, 30]). The Laplace exponents of  $V$  and  $X$  are related by

$$\psi_X(u) = u\psi'_V(u), \quad u > 0 \quad (5.2)$$

(e.g. [2, Remark 4.3]; alternatively, (5.2) can be deduced from (4.1)). Denoting the drifts of  $V$  and  $X$  by  $b_V$  and  $b_X$ , respectively, it is easy to see that

$$b_V = b_X \int_0^\infty e^{-t} dt = b_X. \quad (5.3)$$

Since the negative of the Laplace exponent of any infinitely divisible positive distribution is a Bernstein function and these are concave (cf. [28, Definition 3.1 and Theorem 3.2]) it holds  $u\psi'(u) \geq \psi(u)$  for any such Laplace exponent. Together with the above we observe that

$$\psi_X(u) \geq \psi_V(u) \quad \text{and hence} \quad \int_{(0,\infty)} (1 - e^{-ut}) \nu_X(dt) \leq \int_{(0,\infty)} (1 - e^{-ut}) \nu_V(dt), \quad \forall u \geq 0.$$

Finally, the Lévy density  $x^{-1}k(x)$  of  $V$  with  $k$  non-increasing and the Lévy measure  $\nu_X$  are related by

$$k(x) = \nu_X((x, \infty)), \quad x > 0 \quad (5.4)$$

(e.g. [2, Equation (4.17)]). In particular, the condition  $k(0+) < \infty$  is equivalent to  $\nu_X(\mathbb{R}_+) < \infty$ , and the derivative of  $-k$  is the Lévy density of  $\nu_X$ .

### Differential equation, necessary conditions, and nested ranges

In the next result we give the differential equation for the Laplace transform of  $V$ , which has to be satisfied if  $\mathcal{L}(V)$  is in the range  $D_\xi^+$ . In the special case when  $\eta$  is a compound Poisson process with non-negative jumps, this differential equation (5.5) below has already been obtained by Nilsen and Paulsen [23, Proposition 2]. We then rewrite this differential equation in terms of  $\psi_X$ , which turns out to be very useful for the further investigations.

**Theorem 5.1.** *Let  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ ,  $\sigma, a > 0$  for some standard Brownian motion  $(B_t)_{t \geq 0}$ . Let  $\mu = \mathcal{L}(V) \in L(\mathbb{R}_+)$  have drift  $b_V$  and Lévy density given by  $x^{-1}k(x)$ ,  $x > 0$ , where  $k : (0, \infty) \rightarrow [0, \infty)$  is non-increasing. Then the following are true:*

(i)  $\mu \in R_\xi^+$  if and only if there is some subordinator  $\eta$  such that

$$\frac{1}{2}\sigma^2 u^2 \mathbb{L}_V''(u) + \left(\frac{\sigma^2}{2} - a\right) u \mathbb{L}_V'(u) + \psi_\eta(u) \mathbb{L}_V(u) = 0, \quad u > 0, \quad (5.5)$$

in which case  $\mu = \mathcal{L}(V) = \Phi_\xi(\mathcal{L}(\eta_1))$ . In particular, if  $\eta$  is a subordinator, then the Laplace transform of  $V$  satisfies (5.5) with  $\mathbb{L}_V(0) = 1$ , and if  $V$  is not constant 0, then  $\lim_{u \rightarrow \infty} \mathbb{L}_V(u) = 0$ .

(ii) Let the subordinator  $X = X(\mu)$  be related to  $\mu$  by (5.1). Then  $\mu \in R_\xi^+$  if and only if the function

$$(0, \infty) \rightarrow \mathbb{R}, \quad u \mapsto a\psi_X(u) - \frac{\sigma^2}{2}u\psi_X'(u) - \frac{\sigma^2}{2}(\psi_X(u))^2$$

defines the Laplace exponent  $\psi_\eta(u)$  of some subordinator  $\eta$ . In that case

$$\Phi_\xi(\mathcal{L}(\eta_1)) = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right) = \mu. \quad (5.6)$$

*Proof.* (i) By Theorem 4.1,  $\mu = \mathcal{L}(V) \in R_\xi^+$  if and only if

$$\psi_\eta(u) = \left(a - \frac{\sigma^2}{2}\right) u \psi'_V(u) - \frac{\sigma^2}{2} u^2 (\psi''_V(u) + (\psi'_V(u))^2), \quad u > 0, \quad (5.7)$$

for some subordinator  $\eta$ , in which case  $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$ . Using (4.3) and (4.4), it is easy to see that this is equivalent to (5.5). That  $\mathbb{L}_V(0) = 1$  is clear. If  $V$  is not constant 0, then it cannot have an atom at 0 (e.g. [7, Theorem 2.2]), hence  $\lim_{u \rightarrow \infty} \mathbb{L}_V(u) = 0$ .

(ii) If  $\mathcal{L}(V) = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right) \in L(\mathbb{R}_+)$  for some subordinator  $X$ , then by (5.2)  $\psi'_V(u) = u^{-1} \psi_X(u)$  and  $\psi''_V(u) = u^{-1} \psi'_X(u) - u^{-2} \psi_X(u)$ . Inserting this into (5.7) yields the condition

$$\psi_\eta(u) = a \psi_X(u) - \frac{\sigma^2}{2} u \psi'_X(u) - \frac{\sigma^2}{2} (\psi_X(u))^2, \quad u > 0, \quad (5.8)$$

which gives the claim.  $\square$

**Remark 5.2.** (i) Since  $u \psi'_X(u) \geq \psi_X(u)$  as observed after Equation (5.3), it follows from (5.8) that

$$\psi_\eta(u) \leq \left(a - \frac{\sigma^2}{2}\right) \psi_X(u) - \frac{\sigma^2}{2} (\psi_X(u))^2, \quad u > 0,$$

when the subordinators  $X$  and  $\eta$  are related by (5.6).

(ii) Equation (5.8) is a Riccati equation for  $\psi_X$ . Using the transformation  $y(u) = \exp\left(\int_1^u \frac{\psi_X(v)}{v} dv\right) = C \mathbb{L}_V(u)$  for  $u > 0$  by (5.2), it is easy to see that it reduces to the linear equation (5.5). Unfortunately, in general it is not possible to solve (5.5) in a closed form.

(iii) Since for any subordinator  $\eta$ ,  $\psi_\eta(u)$  has a continuous continuation to  $\{z \in \mathbb{C} : \Re(z) \geq 0\}$  which is analytic in  $\{z \in \mathbb{C} : \Re(z) > 0\}$  (e.g. [28, Proposition 3.6]), for any fixed  $u_0 > 0$  Equation (5.5) can be solved in principle on  $(0, 2u_0)$  by the power series method (e.g. [11, Section 2.8, Theorem 7, p. 190]). In particular when  $\nu_\eta$  is such that  $\int_{(1,\infty)} e^{ux} \nu_\eta(dx) < \infty$  for every  $u > 0$  (e.g. if  $\nu_\eta$  has compact support), then  $\psi_\eta(z) = -b_\eta z + \int_{(0,\infty)} (e^{-zx} - 1) \nu_\eta(dx)$ ,  $z \in \mathbb{C}$ , is an analytic continuation of  $\psi_\eta$  in the complex plane. Hence it admits a power series expansion of the form  $\psi_\eta(z) = \sum_{n=0}^\infty f_n z^n$ ,  $z \in \mathbb{C}$ , with  $f_0 = 0$  and Equation (5.5) may be solved by the Frobenius method (e.g. [11, Sect. 2.8, Theorem 8, p. 215]). To exemplify this, assume for simplicity that  $2a/\sigma^2$  is not an integer. Equation (5.5) has a weak singularity at 0. Its so-called indicial polynomial is given by

$$r \mapsto r(r-1) + \left(1 - \frac{2a}{\sigma^2}\right) r = r \left(r - \frac{2a}{\sigma^2}\right).$$

The exponents of singularity are the zeros of this polynomial, i.e. 0 and  $2a/\sigma^2$ , and since we have assumed that  $2a/\sigma^2$  is not an integer, the general real solution of (5.5) is given by

$$\mathbb{L}_V(u) = C_1 u^{2a/\sigma^2} \sum_{n=0}^\infty c_n u^n + C_2 \sum_{n=0}^\infty d_n u^n, \quad u > 0,$$

where  $C_1, C_2 \in \mathbb{R}$ ,  $c_0 = d_0 = 1$ , the coefficients  $c_n, d_n$  are defined recursively by

$$c_n := \frac{-1}{n(n + 2a/\sigma^2)} \sum_{k=0}^{n-1} c_k f_{n-k}, \quad d_n = \frac{-1}{n(n - 2a/\sigma^2)} \sum_{k=0}^{n-1} d_k f_{n-k}, \quad n \in \mathbb{N},$$

(e.g. [11, Section 2.8, Equation (14), p. 209]) and the power series  $\sum_{n=0}^{\infty} c_n u^n$  and  $\sum_{n=0}^{\infty} d_n u^n$  converge in  $u \in \mathbb{C}$ . Since  $\mathbb{L}_V(0) = 1$ , we even conclude that  $C_2 = 1$ .

Next, we show that the ranges of  $\Phi_\xi$ , when  $\xi_t = \sigma B_t + at$ , are nested when  $\sigma$  and  $a$  vary over all positive parameters.

**Theorem 5.3.** *Let  $B = (B_t)_{t \geq 0}$  be a standard Brownian motion. For  $a, \sigma > 0$  let  $\xi^{(a, \sigma)} := (\xi_t^{(a, \sigma)})_{t \geq 0} := (\sigma B_t + at)_{t \geq 0}$ . Then  $R_{\xi^{(a, \sigma)}}^+ = R_{\xi^{(a/\sqrt{\sigma}, 1)}}^+$ . Further, for  $a, \sigma, a', \sigma' > 0$  such that  $a/\sqrt{\sigma} \leq a'/\sqrt{\sigma'}$  we have  $R_{\xi^{(a, \sigma)}}^+ \subset R_{\xi^{(a', \sigma')}}^+$ . In particular, for fixed  $\sigma > 0$ , the family  $R_{\xi^{(a, \sigma)}}^+$ ,  $a > 0$ , is nested and non-decreasing in  $a$ , and for fixed  $a > 0$  the family  $R_{\xi^{(a, \sigma)}}^+$ ,  $\sigma > 0$ , is nested and non-increasing in  $\sigma$ .*

*Proof.* Since  $(\sigma B_t + at)_{t \geq 0}$  has the same distribution as  $(B_{t\sqrt{\sigma}} + at)_{t \geq 0}$ , we obtain for a Lévy process  $\eta = (\eta_t)_{t \geq 0}$  such that  $\mathcal{L}(\eta_1) \in D_{\xi^{(a, \sigma)}}$  and  $\eta$  is independent of  $B$ ,

$$\int_0^\infty e^{\sigma B_t + at} d\eta_t \stackrel{d}{=} \int_0^\infty e^{B_{t\sqrt{\sigma}} + at} d\eta_t = \int_0^\infty e^{B_t + (a/\sqrt{\sigma})t} d\eta_{t/\sqrt{\sigma}}.$$

Hence  $\mathcal{L}(\eta_{1/\sqrt{\sigma}}) \in D_{\xi^{(a/\sqrt{\sigma}, 1)}}$  and  $\Phi_{\xi^{(a, \sigma)}}(\mathcal{L}(\eta_1)) = \Phi_{\xi^{(a/\sqrt{\sigma}, 1)}}(\mathcal{L}(\eta_{1/\sqrt{\sigma}}))$ . In particular,  $R_{\xi^{(a, \sigma)}}^+ \subset R_{\xi^{(a/\sqrt{\sigma}, 1)}}^+$ . Similarly,  $R_{\xi^{(a, \sigma)}}^+ \supset R_{\xi^{(a/\sqrt{\sigma}, 1)}}^+$  so that  $R_{\xi^{(a, \sigma)}}^+ = R_{\xi^{(a/\sqrt{\sigma}, 1)}}^+$ . For the second assertion, it is hence sufficient to assume  $\sigma = 1$ . Now if  $a < a'$  and  $\mu \in R_{\xi^{(a, 1)}}^+$ , let the subordinator  $X$  be related to  $\mu$  by (5.1). Then

$$a\psi_X(u) - \frac{1}{2}u\psi'_X(u) - \frac{1}{2}(\psi_X(u))^2 = \psi_\eta(u), \quad u > 0,$$

by Theorem 5.1 (ii), hence

$$a'\psi_X(u) - \frac{1}{2}u\psi'_X(u) - \frac{1}{2}(\psi_X(u))^2 = \psi_\eta(u) + (a' - a)\psi_X(u), \quad u > 0,$$

defines the Laplace exponent of a subordinator by [28, Corollary 3.8 (i)]. Hence  $\mu \in R_{\xi^{(a', 1)}}^+$  again by Theorem 5.1 (ii). The remaining assertions are clear.  $\square$

**Remark 5.4.** Although  $R_{\xi^{(1, \sigma)}}^+ \subset R_{\xi^{(1, \sigma')}}^+$  for  $0 < \sigma' < \sigma$ , and  $\sigma B_t + t$  converges pointwise to  $t$  when  $\sigma \rightarrow 0$ , we do not have  $\bigcup_{\sigma > 0} R_{\xi^{(1, \sigma)}}^+ = R_{\xi_{t=t}}^+ (= L(\mathbb{R}_+))$ . For example, a positive 3/4-stable distribution is in  $L(\mathbb{R}_+)$  but not in  $\bigcup_{\sigma > 0} R_{\xi^{(1, \sigma)}}^+$ , as follows from Example 5.6 or Corollary 5.12 below.

While it is difficult to solve the equations (5.5) and (5.8) for given  $\psi_\eta$ , they still allow to obtain results about the qualitative structure of the range. The following gives a simple necessary condition in terms of the Lévy density  $x^{-1}k(x)$  for  $\mu$  to be in  $R_\xi^+$ , and to calculate the drift  $b_\eta$  of  $(\Phi_\xi^+)^{-1}(\mu)$  when  $\mu \in R_\xi^+$ .

**Theorem 5.5.** *Let  $\xi_t = B_t + at$ ,  $t \geq 0$ , for  $\sigma, a > 0$  and some standard Brownian motion  $B = (B_t)_{t \geq 0}$ . Let  $\mu = \mathcal{L}(V) \in L(\mathbb{R}_+)$  with drift  $b_V$  and Lévy density  $x^{-1}k(x)$ . Let the subordinator  $X$  be related to  $\mu$  by (5.1) and denote its drift by  $b_X$ .*

(i) If  $\mu \in R_\xi^+$ , then  $b_X = 0$  and  $\lim_{u \rightarrow \infty} u^{-1/2} |\psi_X(u)| = \lim_{u \rightarrow \infty} u^{1/2} |\psi'_V(u)|$  exists and is finite. If  $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$  for some subordinator  $\eta$  with drift  $b_\eta$ , then  $b_\eta$  and  $\psi_X$  are related by

$$b_\eta = \frac{\sigma^2}{2} \lim_{u \rightarrow \infty} u^{-1} (\psi_X(u))^2 = \frac{\sigma^2}{2} \lim_{u \rightarrow \infty} u (\psi'_V(u))^2. \quad (5.9)$$

(ii) If  $\mu \in R_\xi^+$  has Lévy density  $x^{-1}k(x)$ , then  $b_V = 0$  and  $\limsup_{x \downarrow 0} x^{-1/2} \int_0^x k(s) ds < \infty$ . In particular, if  $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$  for some subordinator  $\eta$  with drift  $b_\eta$ , then  $b_\eta > 0$  if and only if  $\limsup_{x \downarrow 0} x^{-1/2} \int_0^x k(s) ds > 0$ .

*Proof.* (i) Suppose that  $\mu = \mathcal{L}(V) = \Phi_\xi(\mathcal{L}(\eta_1)) \in R_\xi^+$ . Then  $b_V = 0$  by Lemma 4.3 and hence  $b_X = 0$  by (5.3). Since  $\psi'_X(u) = -\int_{(0,\infty)} e^{-ux} x \nu_X(dx)$  we conclude that  $\lim_{u \rightarrow \infty} \psi'_X(u) = 0$  by dominated convergence. Since  $b_X = 0$  and since  $\lim_{u \rightarrow \infty} u^{-1} \psi_X(u) = -b_X = 0$  and  $\lim_{u \rightarrow \infty} u^{-1} \psi_\eta(u) = -b_\eta$  by [28, Remark 3.3 (iv)], (5.9) as well as the necessity of the stated condition follow from (5.8) and (5.2).

(ii) Since  $k(x) = \nu_X((x, \infty))$  by (5.4), it follows from [28, Lemma 3.4] that

$$\frac{e-1}{e} \leq \frac{|\psi_X(u)|}{u \int_0^{1/u} k(s) ds} \leq 1, \quad u > 0.$$

Hence (ii) is an immediate consequence of (i) and Lemma 4.3.  $\square$

**Example 5.6.** Let  $\xi_t = \sigma B_t + at$  be as in Theorem 5.5. Let  $\mu \in L(\mathbb{R}_+)$  with Lévy density  $x^{-1}k(x)$ . Then  $\int_0^1 k(x) dx < \infty$ . If  $\liminf_{s \downarrow 0} k(s)s^{1/2} = +\infty$ , then  $\liminf_{x \downarrow 0} x^{-1/2} \int_0^x k(s) ds = +\infty$ . Hence  $\mu \notin R_\xi^+$ . In particular, a non-degenerate positive  $\alpha$ -stable distribution with  $\alpha > 1/2$  cannot be in  $R_\xi^+$ . A more detailed result will be given in Corollary 5.12 below.

### Selfdecomposable distributions with $k(0+) < \infty$

In this subsection we specialize to selfdecomposable distributions with  $k(0+) < \infty$  and give a characterization when they are in the range  $R_\xi^+$  for  $\xi$  a Brownian motion with drift.

**Theorem 5.7.** Let  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ ,  $\sigma, a > 0$  for some standard Brownian motion  $(B_t)_{t \geq 0}$ . Let  $\mu = \mathcal{L}(V) \in L(\mathbb{R}_+)$  have drift  $b_V$  and Lévy density  $x^{-1}k(x)$ ,  $x > 0$ , where  $k = k_V : (0, \infty) \rightarrow [0, \infty)$  is non-increasing. Let the subordinator  $X = X(\mu)$  be related to  $\mu$  by (5.1). Assume that  $k(0+) < \infty$ , equivalently that  $\nu_X(\mathbb{R}_+) < \infty$ .

(i) Then  $\mu \in R_\xi^+$  if and only if  $b_X = 0$  and  $\nu_X$  has a density  $g$  on  $(0, \infty)$  such that

$$\lim_{t \rightarrow \infty} tg(t) = \lim_{t \rightarrow 0} tg(t) = 0 \quad (5.10)$$

and such that

$$G : (0, \infty) \rightarrow [0, \infty), \quad t \mapsto (a + \sigma^2 \nu_X(\mathbb{R}_+)) \int_0^t g(v) dv + \frac{\sigma^2}{2} tg(t) - \frac{\sigma^2}{2} \int_0^t (g * g)(v) dv \quad (5.11)$$



is non-decreasing. If these conditions are satisfied, then

$$\Phi_\xi(\mathcal{L}(\eta_1)) = \mu,$$

where  $\eta$  is the subordinator with drift 0 and finite Lévy measure  $\nu_\eta(dx) = dG(x)$ .

(ii) Equivalently,  $\mu = \mathcal{L}(V) \in R_\xi^+$  if and only if  $b_V = 0$  and  $-k : (0, \infty) \rightarrow (-\infty, 0]$  is absolutely continuous with derivative  $g$  on  $(0, \infty)$  satisfying (5.10) and such that  $G$  defined by (5.11) is non-decreasing. In that case,  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$ , where  $\eta$  is a subordinator with drift 0 and finite Lévy measure  $\nu_\eta(dx) = dG(x)$ .

*Proof.* (i) Assume that  $\nu_X(\mathbb{R}_+) < \infty$ . Suppose first that  $\mu \in R_\xi^+$ , and let  $(\eta_t)_{t \geq 0}$  be a subordinator such that  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$ . Then  $b_X = 0$  by Theorem 5.5 (i), and by Theorem 5.1 (ii), we have (5.8) with

$$\psi_\eta(u) = -b_\eta u - \int_{(0, \infty)} (1 - e^{-ut}) \nu_\eta(dt) \quad \text{and} \quad \psi_X(u) = - \int_{(0, \infty)} (1 - e^{-ut}) \nu_X(dt), \quad u \geq 0.$$

Since  $\mathbb{L}_{\nu_X}(u)^2 = \mathbb{L}_{\nu_X * \nu_X}(u)$  and  $(\nu_X * \nu_X)(\mathbb{R}_+) = \nu_X(\mathbb{R}_+)^2$ , where  $\mathbb{L}_{\nu_X}$  denotes the Laplace transform of the finite measure  $\nu_X$ , we conclude

$$\begin{aligned} \psi_X(u)^2 &= \left( \int_{(0, \infty)} (1 - e^{-ut}) \nu_X(dt) \right)^2 \\ &= \nu_X(\mathbb{R}_+)^2 - 2\nu_X(\mathbb{R}_+) \int_{(0, \infty)} e^{-ut} \nu_X(dt) + \int_{(0, \infty)} e^{-ut} (\nu_X * \nu_X)(dt) \\ &= \int_{(0, \infty)} (1 - e^{-ut}) (2\nu_X(\mathbb{R}_+) \nu_X - \nu_X * \nu_X)(dt). \end{aligned}$$

Hence, from (5.8), on the one hand

$$\frac{\sigma^2}{2} u \psi'_X(u) = b_\eta u + \int_{(0, \infty)} (1 - e^{-ut}) \rho_1(dt) - \int_{(0, \infty)} (1 - e^{-ut}) \rho_2(dt), \quad (5.12)$$

where

$$\rho_1 := \nu_\eta + \frac{\sigma^2}{2} \nu_X * \nu_X \quad \text{and} \quad \rho_2 := (a + \sigma^2 \nu_X(\mathbb{R}_+)) \nu_X.$$

On the other hand,  $u \psi'_X(u) = -u \int_{(0, \infty)} e^{-ut} t \nu_X(dt)$ , and rewriting  $\int_{(0, \infty)} (1 - e^{-ut}) \rho_i(dt) = \int_0^\infty u e^{-ut} \rho_i((t, \infty)) dt$  by Fubini's theorem as in [28, Remark 3.3(ii)], (5.12) gives

$$\frac{\sigma^2}{2} u \int_{(0, \infty)} e^{-ut} t \nu_X(dt) = -b_\eta u + u \int_0^\infty e^{-ut} (\rho_2((t, \infty)) - \rho_1((t, \infty))) dt, \quad u > 0.$$

Dividing by  $u$ , the uniqueness theorem for Laplace transforms then shows that  $b_\eta = 0$  and that  $\nu_X$  has a density  $g$ , given by

$$g(t) = \frac{2}{\sigma^2 t} (\rho_2((t, \infty)) - \rho_1((t, \infty))), \quad t > 0. \quad (5.13)$$

From this we conclude that  $\lim_{t \rightarrow \infty} tg(t) = 0$  and that  $\lim_{t \rightarrow 0} tg(t) = \frac{2}{\sigma^2}(\rho_2(\mathbb{R}_+) - \rho_1(\mathbb{R}_+))$  exists in  $[-\infty, \infty)$  since  $\rho_2(\mathbb{R}_+) < \infty$ . But since  $g \geq 0$ , the limit must be in  $[0, \infty)$ , hence  $\rho_1(\mathbb{R}_+) < \infty$  so that  $\nu_\eta(\mathbb{R}_+) < \infty$ , and since  $\int_0^1 \frac{tg(t)}{t} dt = \int_0^1 g(t) dt < \infty$ , we also have  $\lim_{t \rightarrow 0} tg(t) = 0$ . Further, by (5.13), the total variation of  $t \mapsto tg(t)$  over  $(0, \infty)$  is finite. Knowing now that  $\nu_X$  has a density  $g$  with  $\lim_{t \rightarrow \infty} tg(t) = \lim_{t \rightarrow 0} tg(t) = 0$ , we can write using partial integration

$$\begin{aligned} u\psi'_X(u) &= \int_0^\infty \left( \frac{d}{dt} e^{-ut} \right) tg(t) dt = \int_0^\infty tg(t) d(e^{-ut}) \\ &= tg(t)e^{-ut} \Big|_{t=0}^{t=\infty} - \int_0^\infty e^{-ut} d(tg(t)) = \int_0^\infty (1 - e^{-ut}) d(tg(t)). \end{aligned}$$

Inserting this in (5.12), we obtain by uniqueness of the representation of Bernstein functions (cf. [28, Theorem 3.2]) that

$$\frac{\sigma^2}{2} d(tg(t)) = \nu_\eta(dt) + \frac{\sigma^2}{2} (g * g)(t) dt - (a + \sigma^2 \nu_X(\mathbb{R}_+)) g(t) dt,$$

or equivalently

$$\nu_\eta(dt) = (a + \sigma^2 \nu_X(\mathbb{R}_+)) g(t) dt + \frac{\sigma^2}{2} d(tg(t)) - \frac{\sigma^2}{2} (g * g)(t) dt. \quad (5.14)$$

Since  $\nu_\eta$  is a positive (and finite) measure, so is the right-hand side of (5.14), and hence  $G$  is non-decreasing with  $\nu_\eta(dt) = dG(t)$ , finishing the proof of the “only if”-assertion. The converse follows by reversing the calculations above, by defining a subordinator  $\eta$  with drift 0 and Lévy measure  $\nu_\eta(dt) := dG(t)$ , observing that  $t \mapsto tg(t)$  is of finite total variation on  $(0, \infty)$  by (5.10) and (5.11), and then showing that  $\nu_\eta$  satisfies (5.12) and hence that  $\psi_\eta$  satisfies (5.8).

(ii) This follows immediately from (i), (5.3) and (5.4).  $\square$

**Remark 5.8.** Let  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ , with  $\sigma, a > 0$  and  $(B_t)_{t \geq 0}$  a standard Brownian motion.

(i) If  $\mu \in R_\xi^+$  and  $X$  is a subordinator such that (5.1) holds and such that  $\nu_X(\mathbb{R}_+) < \infty$ , then the Lévy density  $g$  of  $\nu_X$  cannot have negative jumps, since by (5.11) this would contradict non-decreasingness of  $G$ .

(ii) Let  $X$  be a subordinator with  $\nu_X(\mathbb{R}_+) < \infty$  and  $b_X = 0$ , and suppose that  $\nu_X$  has a density  $g$  such that there is  $r \geq 0$  with  $g(t) = 0$  for  $t \in (0, r]$  and  $g$  is differentiable on  $(r, \infty)$  (the case  $r = 0$  is allowed). Then  $\mathcal{L}(\int_0^\infty e^{-t} dX_t) \in R_\xi^+$  if and only if  $g$  satisfies (5.10) and

$$\left( a + \sigma^2 \nu_X(\mathbb{R}_+) + \frac{\sigma^2}{2} \right) g(t) + \frac{\sigma^2}{2} tg'(t) - \frac{\sigma^2}{2} (g * g)(t) \geq 0, \quad \forall t > r. \quad (5.15)$$

This follows immediately from Theorem 5.1 (iii) since the right-hand side of (5.15) is the derivative of the function  $G$  defined by (5.11).

The following gives an example for a distribution in  $R_\xi^+$  such that  $\nu_X(\mathbb{R}_+) < \infty$ .

**Example 5.9.** Let  $r \geq 0$  and let  $g : [0, \infty) \rightarrow [0, \infty)$  be a function such that  $g(t) = 0$  for all  $t \in (0, r)$  (a void assumption if  $r = 0$ ),  $g|_{[r, \infty)}$  is continuously differentiable with derivative  $g'$ , such that  $g$  is strictly positive on  $[r, \infty)$ ,  $\lim_{t \rightarrow \infty} g(t) = 0$  and such that  $-g'$  is regularly varying at  $\infty$  with index  $\beta < -2$  (in particular,  $g'(t) < 0$  for large enough  $t$ ). Then  $g$  defines a Lévy density of a subordinator  $X$  with drift 0 such that  $\nu_X(\mathbb{R}_+) < \infty$  and  $\mathcal{L}(\int_0^\infty e^{-t} dX_t) \in R_{\sigma B_t + at}^+$  for large enough  $a$ .

*Proof.* Since  $-g'$  is regularly varying with index  $\beta$  and  $\lim_{t \rightarrow \infty} g(t) = 0$ ,  $g$  is regularly varying at  $\infty$  with index  $\beta + 1 < -1$  and  $\lim_{t \rightarrow \infty} \frac{-tg'(t)}{g(t)} = -\beta - 1$  by Karamata's Theorem (e.g. [9, Theorem 1.5.11]). In particular,  $\lim_{t \rightarrow \infty} tg(t) = 0$ , further  $\lim_{t \rightarrow 0} tg(t) = 0$  since  $g(0) < \infty$ , and  $g$  is a density of a finite measure. Next, observe that

$$\frac{(g * g)(t)}{g(t)} = \int_r^{t/2} \frac{g(t-x)}{g(t)} g(x) dx + \int_{t/2}^{t-r} \frac{g(x)}{g(t)} g(t-x) dx, \quad t \geq 2r.$$

But for any  $\varepsilon > 0$ , when  $t \geq t_\varepsilon$  is large enough, we have  $g(t-x)/g(t) \leq 2^{-\beta-1} + \varepsilon$  for  $x \in (r, t/2]$ , and  $g(x)/g(t) \leq 2^{-\beta-1} + \varepsilon$  for  $x \in [t/2, t-r]$  by the uniform convergence theorem for regularly varying functions (e.g. [9, Theorem 1.5.2]). Since  $\int_0^\infty g(t) dt < \infty$ , this shows that  $\limsup_{t \rightarrow \infty} \frac{(g * g)(t)}{g(t)} < \infty$ . Since also  $g * g$  as well as  $|g'|$  are bounded on  $[r, \infty)$ , it follows that (5.15) is satisfied for all  $t \geq r$  for large enough  $a$ , and for  $t \in (0, r)$  it is trivially satisfied. Hence  $\mathcal{L}(e^{-t} dX_t) \in R_\xi^+$  for large enough  $a$ .  $\square$

Next we give some examples of selfdecomposable distributions which are not in  $R_\xi^+$ .

**Example 5.10.** Let  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ , with a standard Brownian motion  $B$  and  $\sigma, a > 0$ .

- (i) A selfdecomposable distribution with Lévy density  $c\mathbb{1}_{(0,1)}(x)x^{-1}$  and  $c > 0$  is not in  $R_\xi^+$  by Theorem 5.7, since  $k(x) = \mathbb{1}_{(0,1)}(x)$  satisfies  $k(0+) < \infty$  but is not continuous.
- (ii) If  $X$  is a subordinator with non-trivial Lévy measure  $\nu_X$  such that  $\nu_X$  has compact support, then  $\mathcal{L}(\int_0^\infty e^{-t} dX_t)$  is not in  $R_\xi^+$  by Theorem 5.7, since if it were then  $\nu_X$  had a density  $g$ , and if  $x_g$  denotes the right end point of the support of  $g$ , then  $2x_g$  is the right endpoint of the support of  $g * g$ , showing that the function  $G$  defined by (5.11) cannot be non-decreasing on  $(0, \infty)$ .
- (iii) If  $X$  is a subordinator with finite Lévy measure and non-trivial Lévy density  $g$  which is a step function (with finitely or infinitely many steps), then  $\mathcal{L}(\int_0^\infty e^{-t} dX_t)$  is not in  $R_\xi^+$  by Remark 5.8 (i), since  $g$  must have at least one negative jump as a consequence of  $\int_0^\infty g(t) dt < \infty$ .

## Positive stable distributions

In this subsection we characterize when a positive stable distribution is in the range  $R_\xi^+$ . We also consider (finite) convolutions of positive stable distributions, i.e. distributions of the form  $\mathcal{L}(\sum_{k=1}^n X_i)$ , where  $n \in \mathbb{N}$  and  $X_1, \dots, X_n$  are independent positive stable distributions.

**Theorem 5.11.** Set  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ ,  $a, \sigma > 0$  for some standard Brownian motion  $(B_t)_{t \geq 0}$ . Let  $0 < \alpha_1 < \dots < \alpha_n < 1$  for some  $n \in \mathbb{N}$  and  $b_i \geq 0$ ,  $i = 1, \dots, n$  and let  $\mu$  be the distribution of  $\sum_{i=1}^n X_i$  where the  $X_i$  are independent and each  $X_i$  is non-trivial and positive  $\alpha_i$ -stable with drift  $b_i$ . Then if  $\mu$  is in  $R_\xi^+$  it holds  $b_i = 0$ ,  $i = 0, \dots, n$ ,  $\alpha_1 \leq (\frac{2a}{\sigma^2} \wedge \frac{1}{2})$  and  $\alpha_n \leq \frac{1}{2}$ . Conversely, if  $b_i = 0$ ,  $i = 0, \dots, n$  and  $\alpha_n \leq (\frac{2a}{\sigma^2} \wedge \frac{1}{2})$ , then  $\mu$  is in  $R_\xi^+$ .

*Proof.* Assume  $\mu = \mathcal{L}(V) = \mathcal{L}(\int_0^\infty e^{-\xi_s} d\eta_s) \in R_\xi^+$  for some subordinator  $\eta$ . Since  $\psi_V(u) = \sum_{i=1}^n \psi_{X_i}(u)$ , the drift of  $V$  is  $\sum_{i=1}^n b_i$ . By Lemma 4.3, this implies  $\sum_{i=1}^n b_i = 0$  and hence  $b_i = 0$  for all  $i$ . Since each  $X_i$  is positive  $\alpha_i$ -stable with drift 0 and non-trivial, we know from [26, Remarks 14.4 and 21.6] that the Laplace exponent of  $X_i$  is given by

$$\psi_{X_i}(u) = \int_{(0, \infty)} (e^{-ux} - 1) \nu_{X_i}(dx) = \int_0^\infty (e^{-ux} - 1) c_i x^{-1-\alpha_i} dx$$

with  $c_i > 0$ . Hence

$$\psi_V(u) = \sum_{i=1}^n \int_0^\infty (e^{-ux} - 1) c_i x^{-1-\alpha_i} dx,$$

such that

$$\psi'_V(u) = - \sum_{i=1}^n c_i u^{\alpha_i-1} \Gamma(1-\alpha_i) \quad \text{and} \quad \psi''_V(u) = \sum_{i=1}^n c_i u^{\alpha_i-2} \Gamma(2-\alpha_i), \quad u > 0.$$

Hence (5.7) reads

$$\begin{aligned} \psi_\eta(u) &= - \sum_{i=1}^n \left[ \left( \left( a - \frac{\sigma^2}{2} \right) c_i \Gamma(1-\alpha_i) + \frac{\sigma^2}{2} c_i \Gamma(2-\alpha_i) \right) u^{\alpha_i} \right. \\ &\quad \left. + \sigma^2 \sum_{j=1}^{i-1} c_i c_j \Gamma(1-\alpha_i) \Gamma(1-\alpha_j) u^{\alpha_i+\alpha_j} + \frac{\sigma^2}{2} c_i^2 (\Gamma(1-\alpha_i))^2 u^{2\alpha_i} \right] \\ &=: - \sum_{i=1}^n \left( A_i u^{\alpha_i} + \sum_{j=1}^{i-1} B_{i,j} u^{\alpha_i+\alpha_j} + C_i u^{2\alpha_i} \right) =: -f(u), \quad u > 0. \end{aligned} \quad (5.16)$$

Observe that  $A_i \in \mathbb{R}$ , and  $B_{i,j}, C_i > 0$  for all  $i, j$ . As the left hand side of (5.16) is the Laplace exponent of a subordinator it is the negative of a Bernstein function [28, Theorem 3.2] and thus  $f(u)$ ,  $u \geq 0$ , has to be a Bernstein function if a solution to (5.16) exists. By [28, Corollary 3.8 (viii)] a Bernstein function cannot grow faster than linearly, which yields directly that  $\alpha_i \in (0, 1/2]$ ,  $i = 1, \dots, n$ . As by [28, Definition 3.1] the first derivative of a Bernstein function is completely monotone, considering  $\lim_{u \rightarrow 0} f'(u) \geq 0$  we further conclude that necessarily  $A_1 \geq 0$ , which is equivalent to  $\alpha_1 \leq \frac{2a}{\sigma^2}$ .

Conversely, let  $V$  be a non-trivial finite convolution of positive  $\alpha_i$ -stable distributions with drift 0 and  $0 < \alpha_1 < \dots < \alpha_n \leq (\frac{2a}{\sigma^2} \wedge \frac{1}{2})$ . Then  $A_i \geq 0$  for all  $i$  and the preceding calculations show that the right hand side of (5.7) is given by  $f(u)$ , which is the Laplace exponent of a subordinator, namely an independent sum of positive  $\alpha_i$ -stable

subordinators (for each  $A_i \geq 0$ ),  $(\alpha_i + \alpha_j)$ -stable subordinators (for each  $B_{i,j}$ ),  $2\alpha_i$ -stable subordinators (for each  $\alpha_i < \frac{1}{2}$ ) and possibly a deterministic subordinator (if  $\alpha_n = 1/2$ ). Hence  $\mathcal{L}(V) \in R_\xi^+$  by Theorem 4.1.  $\square$

As a consequence of the above theorem, we can characterize which positive  $\alpha$ -stable distributions are in  $R_\xi^+$ :

**Corollary 5.12.** *Let  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ ,  $a, \sigma > 0$  for some standard Brownian motion  $(B_t)_{t \geq 0}$ . Then a non-degenerate positive  $\alpha$ -stable distribution  $\mu$  is in  $R_\xi^+$  if and only if its drift is 0 and  $\alpha \in (0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]$ . If this condition is satisfied and  $\mu$  has Lévy density  $x \mapsto cx^{-1-\alpha}$  on  $(0, \infty)$  with  $c > 0$ , then  $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$ , where in the case  $\alpha < 1/2$ ,  $\eta$  is a subordinator with drift 0 and Lévy density on  $(0, \infty)$  given by*

$$x \mapsto c\alpha \left( a - \frac{\sigma^2}{2}\alpha \right) x^{-\alpha-1} + \sigma^2 c^2 \frac{\alpha(\Gamma(1-\alpha))^2}{\Gamma(1-2\alpha)} x^{-2\alpha-1},$$

and in the case  $\alpha = 1/2 = 2a/\sigma^2$ ,  $\eta$  is a deterministic subordinator with drift  $\sigma^2 c^2 (\Gamma(1-\alpha))^2/2$ .

*Proof.* The equivalence is immediate from Theorem 5.11. Further, by (5.16), we have  $\Phi_\xi(\mathcal{L}(\eta_1)) = \mu$  where the Laplace exponent of  $\eta$  is given by

$$\psi_\eta(u) = - \left( \left( a - \frac{\sigma^2}{2} \right) c \Gamma(1-\alpha) + \frac{\sigma^2}{2} c \Gamma(2-\alpha) \right) u^\alpha - \frac{\sigma^2}{2} c^2 (\Gamma(1-\alpha))^2 u^{2\alpha}.$$

The case  $\alpha = 1/2 = 2a/\sigma^2$  now follows immediately, and for  $\alpha < 1/2$  observe that

$$\begin{aligned} \int_0^\infty (e^{-ux} - 1)x^{-1-\beta} dx &= \int_0^u \left( \frac{d}{dv} \int_0^\infty (e^{-vx} - 1)x^{-1-\beta} dx \right) dv \\ &= - \int_0^u v^{\beta-1} \Gamma(1-\beta) dv = - \frac{\Gamma(1-\beta)}{\beta} u^\beta \end{aligned}$$

for  $\beta \in (0, 1)$  und  $u > 0$ , which gives the desired form of the drift and Lévy density of  $\eta$  also in this case.  $\square$

**Example 5.13.** Reconsider Example 1.1, namely,

$$V = \int_0^\infty e^{-(\sigma B_t + at)} dt \stackrel{d}{=} \frac{2}{\sigma^2 \Gamma_{\frac{2a}{\sigma^2}}},$$

where  $V$  has the law of a scaled inverse Gamma distributed random variable with parameter  $\frac{2a}{\sigma^2}$ . In the case that  $\frac{2a}{\sigma^2} = \frac{1}{2}$ , or equivalently  $a = \sigma^2/4$  this is a so called Lévy distribution and it is  $1/2$ -stable (cf. [29, p. 507]). Reassuringly, by Corollary 5.12,  $\mathcal{L}(V)$  is a  $1/2$ -stable distribution if  $a = \sigma^2/4$ .

**Corollary 5.14.** *Let  $\xi_t = \sigma B_t + at$ ,  $t \geq 0$ ,  $\sigma, a > 0$  for some standard Brownian motion  $(B_t)_{t \geq 0}$ . Then  $R_\xi^+$  contains the closure of all finite convolutions of positive  $\alpha$ -stable distributions with drift 0 and  $\alpha \in (0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]$ , which is characterized as the set of infinitely divisible distributions  $\mu$  with Laplace exponent*

$$\psi(u) = \int_{(0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]} m(d\alpha) \int_0^\infty (e^{-ux} - 1) x^{-1-\alpha} dx \quad (5.17)$$

where  $m$  is a measure on  $(0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]$  such that

$$\int_{(0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]} \alpha^{-1} m(d\alpha) < \infty. \quad (5.18)$$

*Proof.* Denote by  $M_1$  the class of all finite convolutions of positive  $\alpha$ -stable distributions with drift 0 and  $\alpha \in (0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]$ , by  $M_2$  its closure with respect to weak convergence, and by  $M_3$  the class of all positive distributions on  $\mathbb{R}$  whose characteristic exponent can be represented in the form (5.17) with  $m$  subject to (5.18). We show that  $M_2 = M_3$ , then since  $M_2 \subset R_\xi^+$  by Theorems 5.11 and 3.4 (i), this implies the statement. To see  $M_2 \subset M_3$ , denote by  $L_\infty(\mathbb{R})$  the closure of all finite convolutions of stable distributions on  $\mathbb{R}$  (cf. [25, Theorem 3.5], where  $L_\infty(\mathbb{R})$  is defined differently, but shown to be equivalent to this definition). Using the fact that  $L_\infty(\mathbb{R})$  is closed, it then follows easily from [25, Theorem 4.1] that also  $M_3$  is closed under weak convergence. Since obviously  $M_1 \subset M_3$  (take  $m$  to be a measure supported on a finite set), we also have  $M_2 \subset M_3$ . Conversely,  $M_3 \subset M_2$  can be shown in complete analogy to the proof of [25, Theorem 3.5].  $\square$

**Remark 5.15.** From the proof of Theorem 5.11 it is possible to obtain a necessary and sufficient condition for a finite convolution of positive, stable distributions to be in  $R_\xi^+$ . Indeed if the  $X_i$  are such that  $\psi_{X_i}(u) = -c_i u^{\alpha_i}$  with  $c_i > 0$  and  $\alpha_i \in (0, 1)$ , then  $\mu = \mathcal{L}(\sum_{i=1}^n X_i)$  is in  $R_\xi^+$  if and only if the function  $f$  defined by (5.16) is a Bernstein function. After ordering the indices, the function  $f$  can be written as  $\sum_{i=1}^m D_i u^{\gamma_i}$  with  $0 < \gamma_1 < \dots < \gamma_m < 2$  and coefficients  $D_i \in \mathbb{R} \setminus \{0\}$ . Since

$$\sum_{i=1, \dots, m; \gamma_i < 1} D_i u^{\gamma_i} = \int_0^\infty (1 - e^{-ux}) \sum_{i=1, \dots, m; \gamma_i < 1} \frac{D_i \gamma_i}{\Gamma(1 - \gamma_i)} x^{-1-\gamma_i} dx$$

as seen in the proof of Corollary 5.12, it follows from [28, Corollary 3.8(viii)] and [26, Example 12.3] that  $f$  is a Bernstein function if and only if  $\gamma_m \leq 1$ ,  $D_m \geq 0$  and

$$\sum_{i=1, \dots, m; \gamma_i < 1} \frac{D_i \gamma_i}{\Gamma(1 - \gamma_i)} x^{-1-\gamma_i} \geq 0, \quad \forall x > 0.$$

## References

- [1] R.B. Ash and C.A. Doléans-Dade (2000). *Probability & Measure*, 2nd edition, Academic Press.

- [2] O.E. Barndorff-Nielsen and N. Shephard (2001). Modelling by Lévy processes for financial econometrics. In: O.E. Barndorff-Nielsen, T. Mikosch, S. Resnick (eds.): *Lévy Processes: Theory and Applications*. Birkhäuser, Boston, pp. 283–318.
- [3] A. Behme (2011). Distributional properties of solutions of  $dV_t = V_{t-}dU_t + dL_t$  with Lévy noise. *Adv. Appl. Prob.* **43**, 688–711.
- [4] A. Behme, A. Lindner and R. Maller (2011). Stationary solutions of the stochastic differential equation  $dV_t = V_{t-}dU_t + dL_t$  with Lévy noise. *Stoch. Proc. Appl.* **121**, 91–108.
- [5] A. Behme and A. Lindner (2013). On exponential functionals of Lévy processes. *J. Theor. Probab.* doi:10.1007/s10959-013-0507-y.
- [6] A. Behme and A. Schnurr (2014+). A criterion for invariant measures of Itô processes based on the symbol. *Submitted*.
- [7] J. Bertoin, A. Lindner and R. Maller (2008). On continuity properties of the law of integrals of Lévy processes. In: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (eds.): *Séminaire de Probabilités XLI, Lecture Notes in Mathematics* **1934**, 137–159, Springer, Berlin.
- [8] J. Bertoin and M. Yor (2005). Exponential functionals of Lévy processes. *Probab. Surveys* **2**, 191–212.
- [9] N.H. Bingham, C.M. Goldie, and J.L. Teugels (1989). *Regular Variation*. Encyclopedia of Mathematics and its Applications, Vol. 27. Cambridge Univ. Press, Cambridge.
- [10] P. Billingsley (1995). *Probability and Measure*. 3rd edition. Wiley Series in Probability and Mathematical Statistics, New York.
- [11] M. Braun (1993). *Differential Equations and Their Applications*. 4th edition. Springer, New York.
- [12] P. Carmona, F. Petit and M. Yor (1997). On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential Functionals and Principal Values Related to Brownian Motion*, Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 73–130.
- [13] R.A. Doney and R.A. Maller (2002). Stability and attraction to normality for Lévy processes at zero and infinity. *J. Theor. Probab.* **15**, 751–792.
- [14] K.B. Erickson and R.A. Maller (2005). Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals. In: M. Emery, M. Ledoux, M. Yor (eds.): *Séminaire de Probabilités XXXVIII, Lecture Notes in Mathematics* **1857**, 70–94, Springer, Berlin.
- [15] H.K. Gjessing and J. Paulsen (1997). Present value distributions with applications to ruin theory and stochastic equations. *Stoch. Proc. Appl.* **71**, 123–144.

- [16] B. Haas and V. Rivero (2012). Quasi-stationary distributions and Yaglom limits of self-similar Markov processes. *Stoch. Proc. Appl.* **122**, 4054–4095.
- [17] Z.J. Jurek and J.D. Mason (1993). *Operator-Limit Distributions in Probability Theory*. Wiley, New York.
- [18] Z. J. Jurek and W. Vervaat (1983). An integral representation for self-decomposable Banach space valued random variables. *Z. Wahrsch. Verw. Gebiete* **62**, 247–262.
- [19] O. Kallenberg (2001). *Foundations of Modern Probability*. 2nd edition, Springer, Berlin.
- [20] J.G. Llavona (1986). *Approximation of Continuously Differentiable Functions*. Mathematics Studies, Vol. 130, North-Holland, Amsterdam.
- [21] T. Liggett (2010). *Continuous Time Markov Processes. An Introduction*. AMS Graduate Studies in Mathematics 113. Providence, RI.
- [22] A. Lindner and R. Maller (2005). Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. *Stoch. Proc. Appl.* **115**, 1701–1722
- [23] T. Nilsen and J. Paulsen (1996). On the distribution of a randomly discounted compound Poisson process. *Stoch. Proc. Appl.* **61**, 305–310.
- [24] J.C. Pardo, V. Rivero and K. van Schaik (2014+). On the density of exponential functionals of Lévy processes. *Bernoulli*, to appear.
- [25] K. Sato (1980). Class  $L$  of multivariate distributions and its subclasses. *J. Multivar. Anal.* **10**, 207–232.
- [26] K. Sato (2013). *Lévy Processes and Infinitely Divisible Distributions*. Revised Edition. Cambridge University Press, Cambridge.
- [27] K. Sato (2007). Transformations of infinitely divisible distributions via improper stochastic integrals. *ALEA* **3**, 67–110.
- [28] R. L. Schilling, R. Song and Z. Vondracek (2012). *Bernstein Functions. Theory and Applications*. 2nd edition. De Gruyter Studies in Mathematics 37. Berlin: Walter de Gruyter.
- [29] F. W. Steutel and K. van Harn (2003). *Infinite Divisibility of Probability Distributions on the Real Line*. Marcel Dekker Inc, New York.
- [30] S. J. Wolfe (1982). On a continuous analogue of the stochastic difference equation  $X_n = \rho X_{n-1} + B_n$ . *Stoch. Proc. Appl.* **12**, 301–312.